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Transverse Particle Equations of Motion

Outline:

§1 Particle Equations of Motion

- Derivation of transverse equations
 - Basic form
 - Including bending and dispersive terms

§2 Transverse Particle Equations of Motion in Linear Focusing Channels

- Continuous Focusing
- Quadrupole Focusing
- Solenoidal Focusing

§3 Description of Applied Focusing Fields

- Overview
- Multipole descriptions

§4 Transverse Particle Equations of Motion with Nonlinear Applied Fields

- Approach 1: Explicit 3D
- Approach 2: Perturbed linear

§5 Linear Equations of Motion without Space-Charge, Acceleration, and Momentum Spread

- Hill's Equation
- Orbit stability and eigenvalue structure

§6 Floquet's Theorem and the Phase-Amplitude form of the Particle Orbit.

- Floquet's theorem
- Phase amplitude form of the particle orbit
- Particle phase advance.

§7 The Courant-Snyder Invariant and Single-Particle Emittance

- Derivation - Courant Snyder Invariant.
- Interpretation and uses.

§8 The Betatron Formulation of the Particle Orbit

- Formulation
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§9 Momentum Spread Effects

- Overview and equations
- Dispersion function

§10 Acceleration and Normalized Emittance.

- Transformation of orbit equations to standard form
- Normalized emittance.

References

§1 Particle Equations of Motion

The Lorentz force on a particle of charge q and mass m at coordinate $\vec{x}(t)$ is given by: (SI units)

$$\frac{d}{dt} \vec{p}_i(t) = q \left[\vec{E}(\vec{x}_i, t) + \vec{v}_i(t) \times \vec{B}(\vec{x}_i, t) \right] = F_i$$

$i = \text{particle index}$
 $t = \text{time}$

$$\vec{p}_i(t) = m \gamma_i(t) \vec{v}_i(t)$$

particle momentum

$$\vec{v}_i(t) = c \vec{\beta}_i(t)$$

particle velocity, $c =$

$$\gamma_i(t) = \frac{1}{\sqrt{1 - \beta_i^2(t)}}$$

speed of light
 in vacuum

where

	Total-Field	Applied-Field	Self-Field
Electric Field:	$\vec{E}(\vec{x}, t) = \vec{E}_a(\vec{x}, t) + \vec{E}_s(\vec{x}, t)$		
Magnetic Field:	$\vec{B}(\vec{x}, t) = \vec{B}_a(\vec{x}, t) + \vec{B}_s(\vec{x}, t)$		

Applied Field sources:

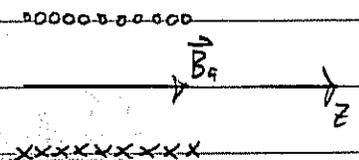
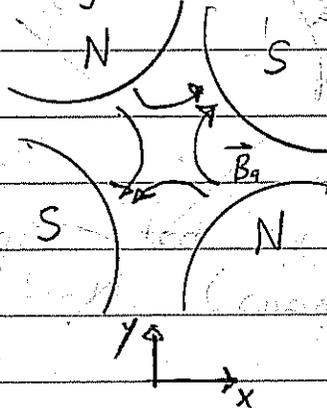
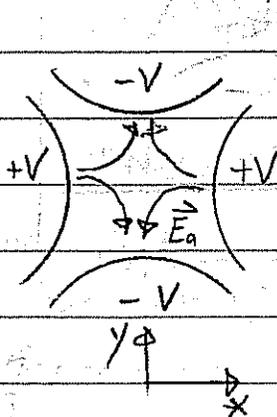
- Focusing Optics
- Dipole Bends
- Accelerating Fields

Focusing Optics (Examples)

Electric Quadrupole

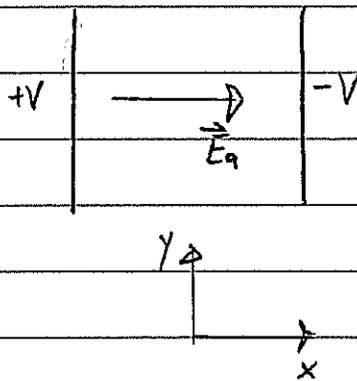
Magnetic Quadrupole

Solenoid

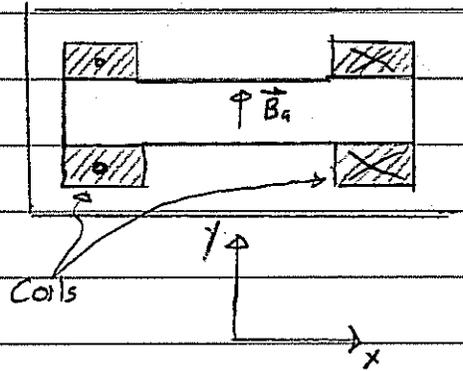


Dipole Bends (Examples)

Electric Dipole
x-direction bend



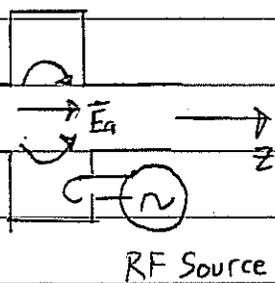
Magnetic Dipole
x-direction bend.



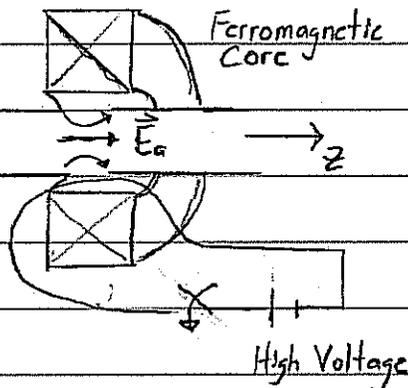
Bend
is typically
x-plane

Accelerating Fields (Examples)

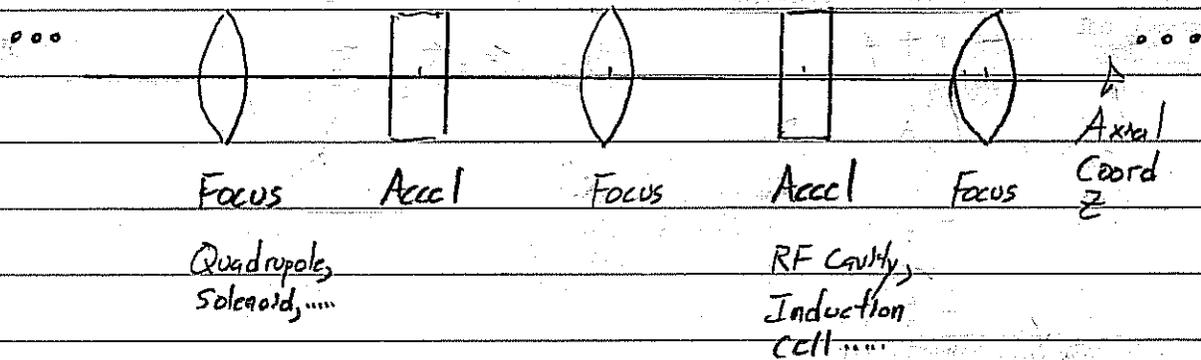
RF Cavity



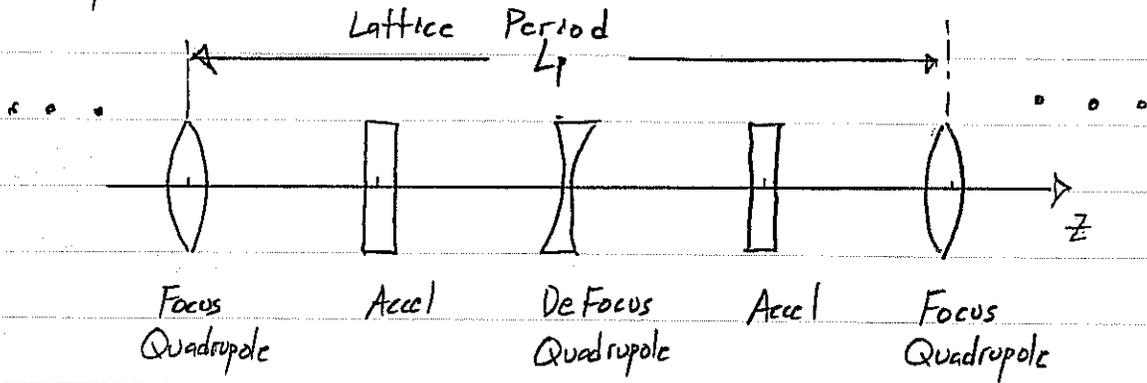
Induction Cell



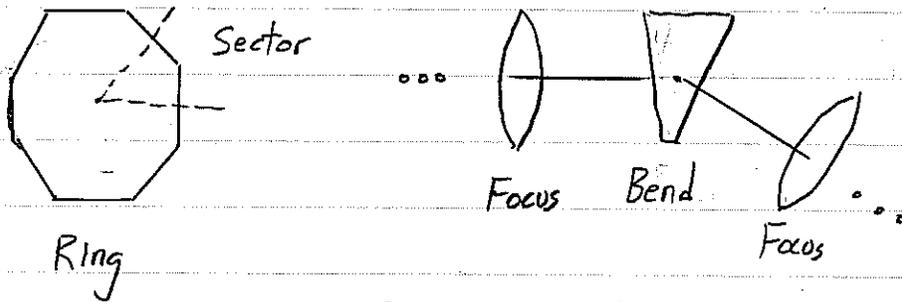
Applied field structures are often arranged in a regular (periodic) lattice for transport/acceleration.



Example - Linear FODO Lattice:



Some lattices for rings may also incorporate bends and more complicated periodic focusing structures.

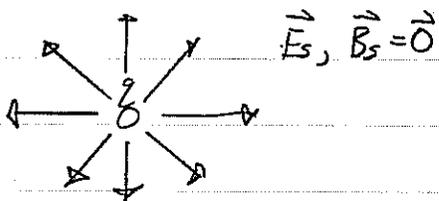


Focus usually 3 or more Focus/DeFocus Quadrupoles (Triplet etc).

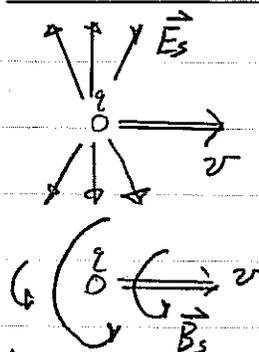
Self-Fields

Are generated by the distribution of beam particles
 • charges
 • currents

Particle at Rest



Particle in Motion



(Relativistic)
 See Jackson, "Classical Electrodynamics"

"Boost" Lab-Frame E_s field to moving frame

- Superimpose for particles in beam distribution
- Accelerating particles also radiate.

\vec{E} and \vec{B} satisfy the Maxwell Equations, and the linear structure of the Maxwell Equations can be exploited to express:

Applied Fields (often quasi-static)

$$\begin{aligned} \nabla \cdot \vec{E}_a &= \frac{\rho_a}{\epsilon_0} & \nabla \times \vec{B}_a &= \mu_0 \vec{J}_a + \frac{1}{c^2} \frac{d\vec{E}_a}{dt} \\ \nabla \times \vec{E}_a &= -\frac{d\vec{B}_a}{dt} & \nabla \cdot \vec{B}_a &= 0 \end{aligned}$$

ρ_a = applied charge density
 \vec{J}_a = applied current density

$\frac{1}{\mu_0 \epsilon_0} = c^2$

+ Boundary Conditions on \vec{E}_a, \vec{B}_a

Self Fields

$$\begin{aligned} \nabla \cdot \vec{E}_s &= \sum_{i=1}^N q_i \frac{\delta(\vec{x} - \vec{x}_i(t))}{\epsilon_0} & \nabla \times \vec{B}_s &= \mu_0 \sum_{i=1}^N q_i \vec{v}_i \delta(\vec{x} - \vec{x}_i(t)) \\ & & & + \frac{1}{c^2} \frac{d\vec{E}_s}{dt} \\ \nabla \times \vec{E}_s &= -\frac{d\vec{B}_s}{dt} & \nabla \cdot \vec{B}_s &= 0 \end{aligned}$$

$\sum_{i=1}^N$ = sum over all N particles in beam

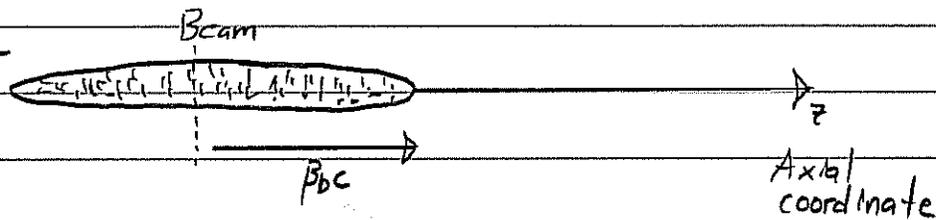
$\delta(\vec{x})$ = Dirac-delta function

+ Boundary Conditions on \vec{E}_s, \vec{B}_s

from material structures, radiation conditions etc.

In accelerator beams, there is ideally a single species of ion, many particles, and the motion of the "bunch" is highly directed:

Directed Motion



ion velocity

$$\frac{d\vec{x}_i}{dt} = \vec{v}_i = \beta_{bc} \hat{z} + \delta\vec{v}_i \quad \beta_{bc} = \text{Mean axial Relativistic factor of beam}$$

$$\underline{\underline{|\delta\vec{v}_i| \ll \beta_{bc}}} \Rightarrow \underline{\underline{\text{Paraxial Approx.}}}$$

Many Particles

$$\underline{\underline{\sum_i q_i \delta(\vec{x} - \vec{x}_i(t)) \approx \rho(\vec{x}, t)}}} \quad \text{continuous charge density}$$

$$\underline{\underline{\sum_i q_i \vec{v}_i \delta(\vec{x} - \vec{x}_i(t)) \approx \beta_{bc} \rho(\vec{x}, t) \hat{z}}} \quad \text{continuous axial current density}$$

Moreover, the evolution is typically sufficiently slow for heavy ions where in the Maxwell equations we can neglect radiation and approximate the self-field Maxwell equations (see Intro. lecture by J.J. Barnard, Electrostatic approximation) as:

$$\begin{aligned} \vec{E}_s &= -\nabla\phi \\ \vec{B}_s &= \nabla \times \vec{A}, \quad \vec{A} = \frac{\beta_{bc}}{c} \phi \hat{z} \end{aligned}$$

and Boundary conditions on ϕ

$$\nabla^2 \phi = \frac{\partial}{\partial x} \cdot \frac{\partial}{\partial x} \phi = -\frac{\rho}{\epsilon_0}$$

The force acting on the beam particles can be resolved into applied and self-field components:

$$\vec{F}_i(\vec{x}_i, t) = q \vec{E}(\vec{x}_i, t) + q \vec{v}_i(t) \times \vec{B}(\vec{x}_i, t)$$

$$\begin{aligned} \vec{E} &= \vec{E}_a + \vec{E}_s \\ \vec{B} &= \vec{B}_a + \vec{B}_s \end{aligned}$$

Applied:

$$\vec{F}_a = q \vec{E}_a + q \vec{v}_i \times \vec{B}_a$$

Self:

$$\vec{F}_s = q \vec{E}_s + q \vec{v}_i \times \vec{B}_s$$

The self-field force can then be expressed as (see also J.J. Barnard, Intro Lectures):

$$\vec{F}_s = q (\vec{E}_s + \vec{v}_i \times \vec{B}_s)$$

$$\approx q \left[\underbrace{-\nabla\phi}_{\text{Transverse and Longitudinal}} + \beta_b c \cdot \hat{z} \times \underbrace{(-\nabla \times \frac{\beta_b}{c} \phi \hat{z})}_{\text{Transverse Only}} \right]$$

$$\vec{F}_s \approx -q (1 - \beta_b^2) \nabla_{\perp} \phi - q \frac{\partial \phi}{\partial z} \hat{z}$$

$$\vec{F}_s \approx -q \frac{1}{\gamma_b^2} \nabla_{\perp} \phi - q \frac{\partial \phi}{\partial z} \hat{z}$$

where

$$\gamma_b \equiv \frac{1}{\sqrt{1 - \beta_b^2}}$$

is the axial relativistic factor of the beam

The particle equations of motion become:

Transverse:

$$\frac{d}{dt} (m \gamma_i \vec{v}_{\perp i}) \approx \underbrace{q \vec{E}_{\perp}^a + q \beta_{0c} \hat{z} \times \vec{B}_{\perp}^a + q B_z^a \vec{v}_{\perp i} \times \hat{z}}_{\text{applied terms}} - \underbrace{\frac{q}{\gamma_b^2} \nabla_{\perp} \phi}_{\text{self-term}}$$

Longitudinal

$$\frac{d}{dt} (m \gamma_i v_{zi}) \approx \underbrace{q E_z^a - q (v_{xi} B_y^a - v_{yi} B_x^a)}_{\text{Applied terms}} - \underbrace{q \frac{\partial \phi}{\partial z}}_{\text{self-term}}$$

In the remainder of this lecture, we only analyze transverse dynamics. Longitudinal dynamics will be covered in lectures by J. J. Barnard.

- Except near injector, acceleration is typically slow
 - Fractional changes in β_b and γ_b are small over characteristic lattice period of a periodic accelerator lattice.
- Can often regard γ_b , β_b as specified functions of s , given by the "acceleration schedule"

In transverse accelerator dynamics, it is convenient to employ the axial coordinate of a slice of beam particles in the accelerator as the independent variable:

$$\beta_b = \sum_{i=1}^N \frac{v_{zi}}{c}$$

slice

$$S \equiv s_i + c \int_{t_i}^t dt \tilde{\beta}_b(\tilde{t})$$

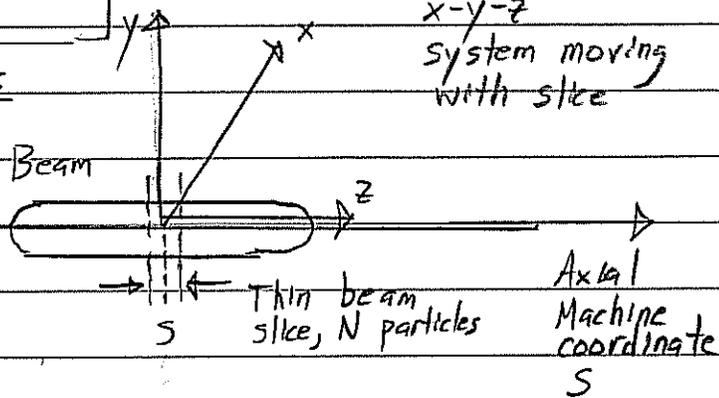
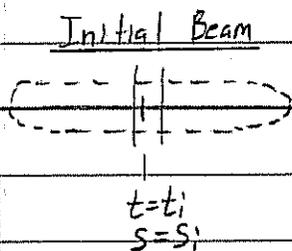
Time t

Notation Comment:

v_{zi} \wedge $i =$ particle index

t_i } initial time, s
 s_i } (same for all particles)

$x-y-z$ system moving with slice



Transform:

$$dt = \frac{ds}{\beta_b} \Rightarrow v_x = \frac{dx}{dt} \approx \beta_b c \frac{dx}{ds} \equiv \beta_b c x'$$

Denote:

etc.

$$' \equiv \frac{d}{ds}$$

$$v_x = \frac{dx}{dt} \approx \beta_b c x'$$

$$v_y = \frac{dy}{dt} \approx \beta_b c y'$$

x', y' angles trajectory makes with z -axis.
 Paraxial Approx:
 $x'^2, y'^2 \ll 1$
 $\Rightarrow \frac{v_x^2}{c^2}, \frac{v_y^2}{c^2} \ll \frac{v_z^2}{c^2}$

Transform transverse particle equation of motion:

$$\frac{d}{dt} \left(m \gamma_i \frac{d\vec{x}_{\perp i}}{dt} \right) = m \gamma_i \frac{d}{ds} \left(\gamma_i \beta_{zi} \frac{d\vec{x}_{\perp i}}{ds} \right)$$

$$= m \gamma_i \beta_{zi}^2 \frac{d^2 \vec{x}_{\perp i}}{ds^2} + m \frac{d\vec{x}_{\perp i}}{ds} \beta_{zi} \frac{d}{ds} \left(\gamma_i \beta_{zi} \right)$$

But:

Term 1

Term 2

$$\text{Term 1: } m \gamma_i \beta_{zi}^2 \frac{d^2 \vec{x}_{\perp i}}{ds^2} \approx m \gamma_b \beta_{bc}^2 \frac{d^2 \vec{x}_{\perp i}}{ds^2} = m \gamma_b \beta_{bc}^2 \vec{x}_{\perp i}''$$

$$\begin{aligned} \text{Term 2: } m \frac{d\vec{x}_{\perp i}}{ds} \beta_{zi} \frac{d}{ds} \left(\gamma_i \beta_{zi} \right) &\approx m \frac{d\vec{x}_{\perp i}}{ds} \beta_{bc} \frac{d}{ds} \left(\gamma_b \beta_{bc} \right) \\ &\approx m \beta_{bc} \left(\gamma_b \beta_{bc} \right)' \vec{x}_{\perp i}' \end{aligned}$$

Summary:

$$\Rightarrow \boxed{m \frac{d}{dt} \left(\gamma_i \frac{d\vec{x}_{\perp i}}{dt} \right) \approx m \gamma_b \beta_{bc}^2 \left[\vec{x}_{\perp i}'' + \frac{1}{\left(\gamma_b \beta_{bc} \right)'} \frac{d}{ds} \left(\gamma_b \beta_{bc} \right) \vec{x}_{\perp i}' \right]}$$

Also, for the applied field component terms:

$$\boxed{g B_z^a \vec{v}_{\perp i} \times \hat{z} = g B_z^a \beta_{bc} \vec{x}_{\perp i}' \times \hat{z}}$$

Use

Using these results the transverse particle equations of motion on pg 7 become:

Drop particle "i" subscripts to clean up notation:

Transverse Particle Equations of Motion

$$\begin{aligned} \vec{X}_\perp'' + \frac{(\gamma_b \beta_b)'}{(\gamma_b \beta_b)} \vec{X}_\perp' &= \frac{q}{m \gamma_b \beta_b c^2} \vec{E}_\perp^a + \frac{q}{m \gamma_b \beta_b c} \hat{z} \times \vec{B}_\perp^a \\ &+ \frac{q B_z^a}{m \gamma_b \beta_b c} \vec{X}_\perp' \times \hat{z} - \frac{q}{\gamma_b^3 \beta_b^2 c} \nabla_\perp \phi \end{aligned}$$

\vec{E}_a = Applied Electric Field

\vec{B}_a = Applied Magnetic Field

$$\nabla^2 \phi = \frac{\partial}{\partial x} \cdot \frac{\partial}{\partial x} \phi = -\frac{\rho}{\epsilon_0}$$

and boundary conditions on ϕ

$$l \equiv \frac{d}{ds}$$

Note:

- γ_b factors different in applied and self-field terms.

- In $-\frac{q}{\gamma_b^3 \beta_b^2 c} \nabla_\perp \phi$ contributions to γ_b^3 come from:

- $\gamma_b \Rightarrow$ kinematics

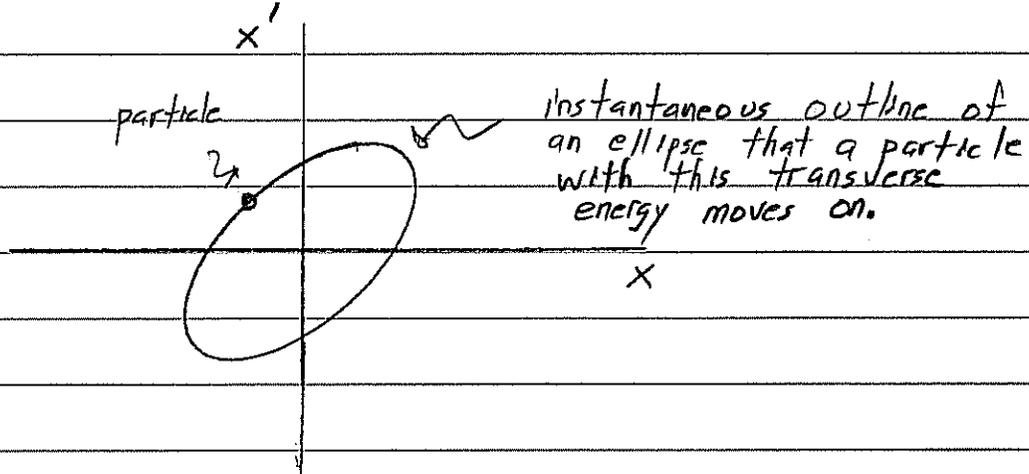
- $\gamma_b^2 \Rightarrow$ leading order self-magnetic field corrections.

Overview:

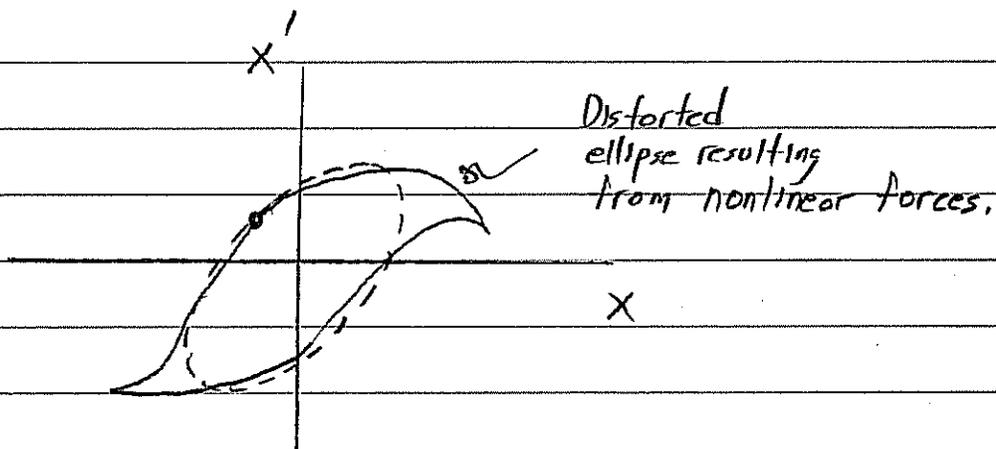
S.M. Lund 11/

Much transverse accelerator physics centers on understanding the evolution of beam particles in 4-dimensional $x-x'$, $y-y'$ phase-space.

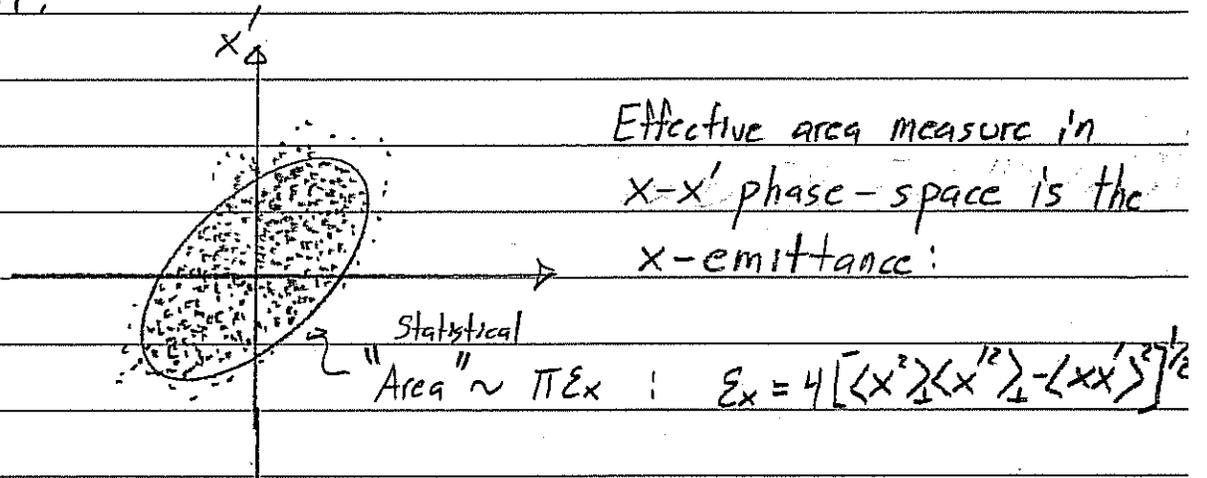
Often phase-space projections in $x-x'$ and $y-y'$ are analyzed



When forces acting on particles are linear, particles tend to move on ellipses of constant area (which may elongate/shrink and rotate) in phase-space. Nonlinear forces can distort orbits and cause undesirable effects, growing effective phase-space area.



The effective phase-space volume of a distribution of beam particles is of fundamental interest.



We will find in statistical beam descriptions:

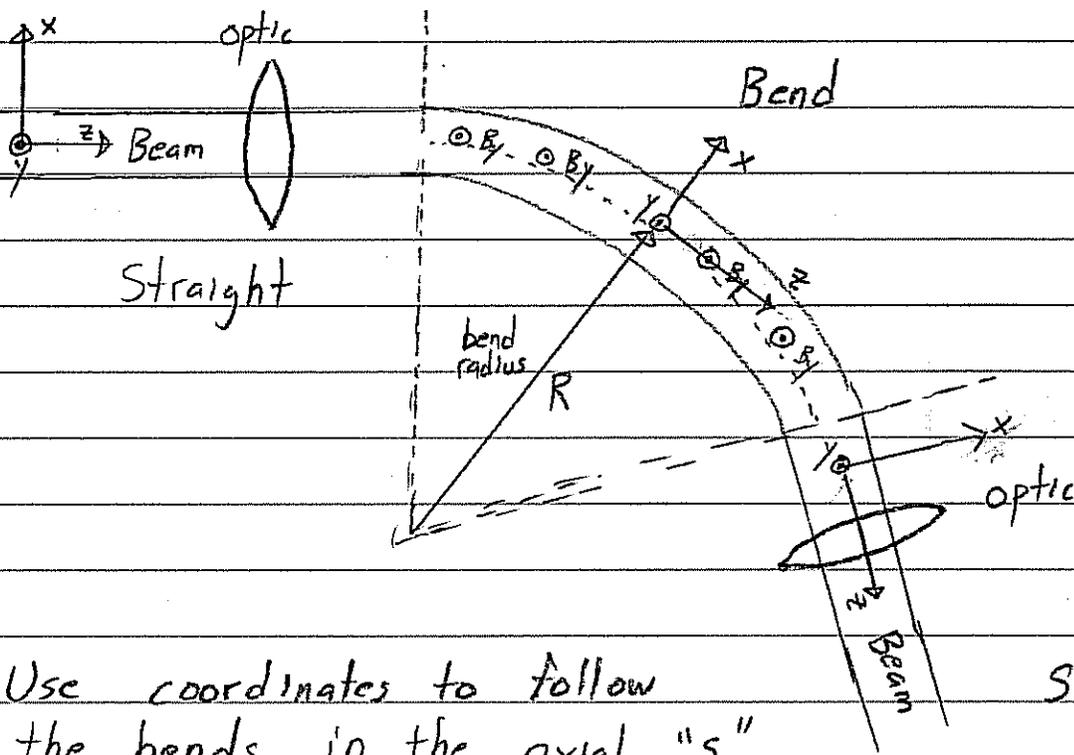
Larger phase-space areas \Rightarrow Harder to achieve
(larger emittances) small beam spots
on target.

Smaller phase-space areas \Rightarrow Easier to achieve
(smaller emittances) small beam spots.
on target

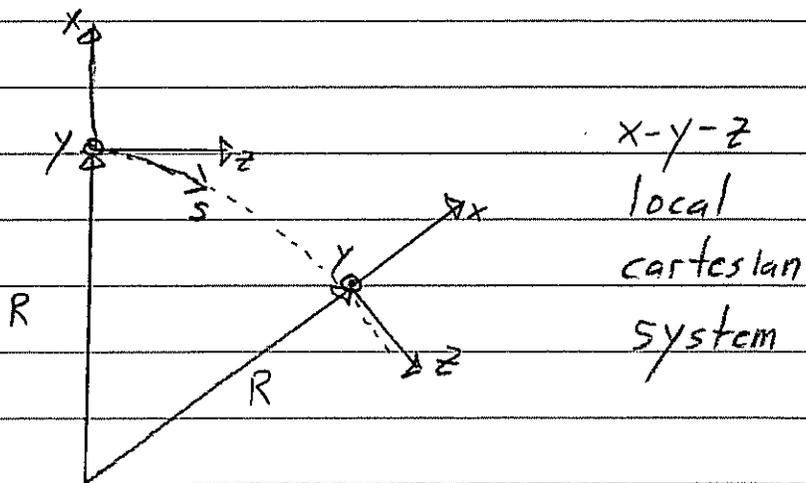
Much of advanced accelerator physics seeks to understand and control emittance growth due to nonlinear applied fields and space-charge effects. In the remainder of this lecture we will review basic accelerator physics for the transverse dynamics of a particle moving in applied fields. Later we will generalize concepts to include space-charge.

Bent Coordinate System to analyze particle equations of motion with dipole bends and spread in axial particle momentum

The previous equations of motion can be applied to dipole bends provided the $x-y-z$ coordinate system is fixed. In practice, it can be more convenient to employ coordinates that follow the beam in a bend.



Use coordinates to follow the bends in the axial "s" coordinate.



Using this system, dipoles are adjusted given the mean beam energy / momentum to bend the required amount for a bend radius R . Usually, bends are only in one plane, say the x -plane, and are implemented by a dipole B_y^a or E_x^a applied field.

Denote:

$$p_0 = m \gamma_b \beta_{bc} = \text{design momentum}$$

Then the x -bending radius = R of the design momentum particle $p_0 = \text{const}$ in a constant applied magnetic field $\vec{B}^a = B_y^a \hat{y} = \text{const}$ is given by

$$\frac{1}{R} = \frac{q B_y^a}{c p_0}$$

$$B_y^a = \text{const}$$

$$R = \text{const.}$$

Often this is expressed in terms of the "particle rigidity" $[B\rho]$ (read as one symbol and called "B-rho").

$$[B\rho] \equiv \frac{c p_0}{q} = \frac{m \gamma_b \beta_{bc}}{q}$$

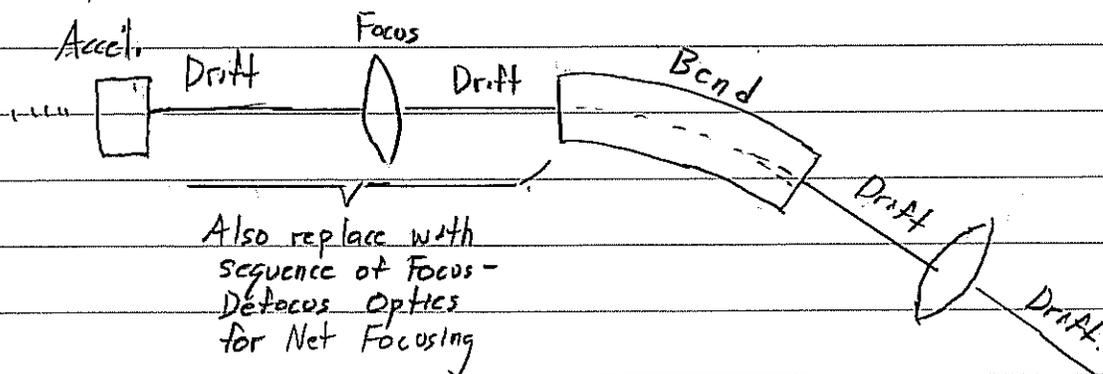
Then

$$\frac{1}{R} = \frac{B_y^a}{[B\rho]}$$

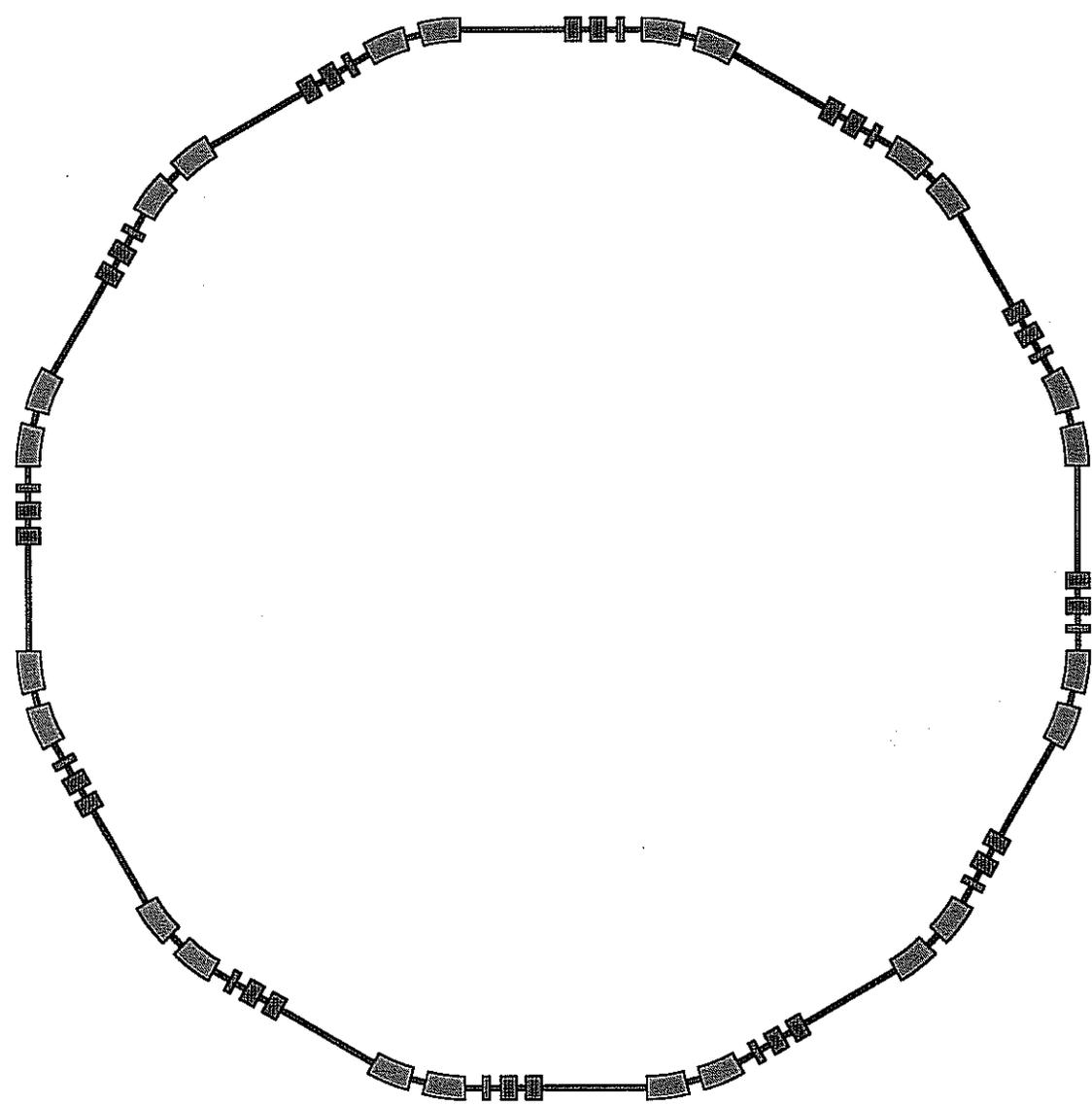
Comment: Different design formulas will result for electric bends with $E_x^a \neq 0$ (see problems)

In straight sections $R \rightarrow \infty$, and bends can be positive or negative radius R (direction of curve) depending on the sign of B_y^g and $[B\rho]$. Practical machines are often made from discrete element dipoles and straight sections with separated function optics. (See the sketch of a practical ring on the next page). For a ring, such dipoles are tuned with the particle rigidity such that the orbit makes a complete lap through the machine. Dipoles must be adjusted as the particles gain energy to maintain a closed orbit in the machine aperture. The need to synchronously adjust bending dipoles and focusing elements with the gains in particle energy is the origin of the name "Synchrotron" for such machines. The total bending strength of a synchrotron in tesla-meters limits ultimate achievable particle energy/momentum in the machine.

Typical Separated Function Lattice Period



SIS – 18 Synchrotron GSI, Germany 18 Tesla–Meter Bending Strength



*Acceleration Cells
Not Shown*

-  Dipole Bending Magnets
-  Quadrupole Focusing Magnets (Triplets)

Off momentum effects:

$$p_s = m \gamma_b \beta_b c = \text{design momentum.}$$

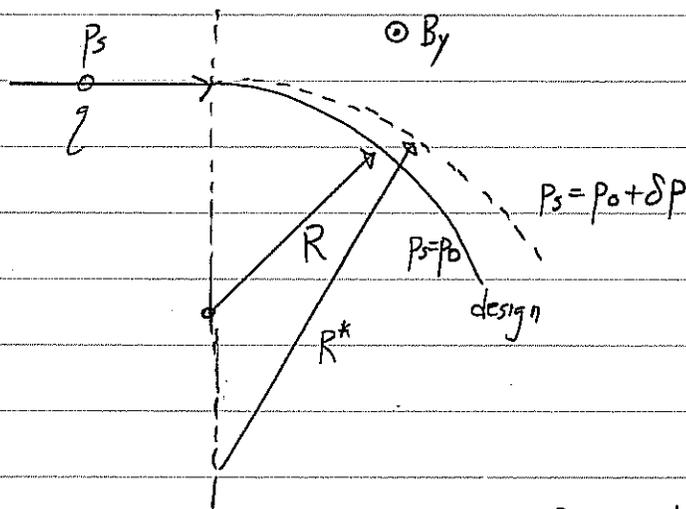
For momentum errors:

$$p_s = p_0 + \delta p$$

$$p_0 = m \gamma_b \beta_b c = \text{"design-momentum"}$$

$$\delta p = \text{"off-momentum"}$$

This will modify the particle equations of motion, particularly in cases where there are bends since particles with different momenta will be bent at different radii.



Comment: It is not common to have acceleration in bends. However, dipole bends and quadrupole focusing magnets are sometimes combined.

Without presenting algebraic detail (see standard accelerator texts such as Edwards and Syphers) the particle equations of motion for a coasting beam with design momentum p_0 become
(the full derivation is beyond the scope of this class).

$$x''(s) + \frac{(\alpha_b \beta_b)'}{(\alpha_b \beta_b)} x'(s) + \left[\frac{1}{R^2(s)} \frac{1-\delta}{1+\delta} \right] x(s) = \frac{\delta}{1+\delta} \frac{1}{R(s)} + \frac{g}{m \alpha_b \beta_b c^2} \frac{E_x^q}{(1+\delta)^2}$$

$$- \frac{g}{m \alpha_b \beta_b c^2} \frac{B_y^q}{1+\delta} + \frac{g B_s^q}{m \alpha_b \beta_b c^2} \frac{y'(s)}{1+\delta} - \frac{g}{m \alpha_b \beta_b c^2} \frac{1}{1+\delta} \frac{d\phi}{dx}$$

$$y''(s) + \frac{(\alpha_b \beta_b)'}{(\alpha_b \beta_b)} y'(s) = \frac{g}{m \alpha_b \beta_b c^2} \frac{E_y^q}{(1+\delta)^2} + \frac{g}{m \alpha_b \beta_b c^2} \frac{B_x^q}{1+\delta}$$

$$- \frac{g B_s^q}{m \alpha_b \beta_b c^2} \frac{x'(s)}{1+\delta} - \frac{g}{m \alpha_b \beta_b c^2} \frac{1}{1+\delta} \frac{d\phi}{dy}$$

$$\delta \equiv \frac{\Delta p}{p_0} = \text{Fractional Momentum Error.}$$

$$p_0 = m \alpha_b \beta_b c = \text{design momentum, } \frac{1}{R} = \frac{B_y^q |_{\text{dipole}}}{[B\rho]}$$

Comments:

$$[B\rho] = p_0 / q$$

- Design bends only in x and E_x^q , B_y^q contain no dipole field terms. These components are set consistent with the design bend radius $R(s)$
- x and y equations differ significantly due to the bends modifying the x -equation when $R(s) \neq \infty$.
- Equations contain only low-order terms in δ .

§2 Transverse Particle Equations of Motion
in Linear^{*} Focusing Channels

* No Bends and
 Linear Applied Fields

The transverse particle equations of motion previously derived in §1 can be expressed as:

$$x'' + \left[\frac{1}{\gamma_b \beta_b} \frac{d}{ds} (\gamma_b \beta_b) \right] x' = \frac{q}{m \gamma_b^3 \beta_b^2 c^2} E_x^a - \frac{q}{m \gamma_b \beta_b c} B_y^a + \frac{q}{m \gamma_b \beta_b c} B_z^a y' - \frac{q}{m \gamma_b^3 \beta_b^2 c^2} \frac{\partial \phi}{\partial x}$$

$$y'' + \left[\frac{1}{\gamma_b \beta_b} \frac{d}{ds} (\gamma_b \beta_b) \right] y' = \frac{q}{m \gamma_b^3 \beta_b^2 c^2} E_y^a + \frac{q}{m \gamma_b \beta_b c} B_x^a - \frac{q}{m \gamma_b \beta_b c} B_z^a x' - \frac{q}{m \gamma_b^3 \beta_b^2 c^2} \frac{\partial \phi}{\partial y}$$

Assumptions:

- Paraxial equations: $x'^2, y'^2 \ll 1$
- Dispersive effects neglected; $v_z = \beta_b c$ for all particles, but acceleration (β_b may vary in s) is allowed.
- Electrostatic and leading-order (in β_b) self-magnetic interactions included.
- Fixed x-y-z coordinate system (no local bends).

To complete the description of the \perp particle equations of motion, the applied focusing fields

electric:	E_x^a, E_y^a, E_z^a
magnetic:	B_x^a, B_y^a, B_z^a

must be specified as a function of s and the \perp particle coordinates x, y . Also the consistent change in axial velocity with β_{0c} must be evaluated due to E_z^a and possibly time-varying electromagnetic fields (as in RF cavities). Here we restrict ourselves to \vec{E}_a, \vec{B}_a resulting from applied focusing structures. In intense beam accelerators and transport lattices, focusing structures are designed to optimize linear focusing forces with:

Linear
Focusing

$$E_x^a \approx (\text{function of } s) \cdot (x \text{ or } y)$$

$$E_y^a \approx (\text{function of } s) \cdot (x \text{ or } y)$$

$$B_x^a \approx (\text{function of } s) \cdot (x \text{ or } y)$$

$$B_y^a \approx (\text{function of } s) \cdot (x \text{ or } y)$$

$$B_z^a \approx (\text{function of } s)$$

We now review some common situations that realize these forms:

- 1/ Continuous Focusing
- 2/ Quadrupole Focusing
- 3/ Solenoidal Focusing

Continuous Focusing

$$\begin{cases} \vec{B}_0^a = 0 \\ E_x^a = -\frac{m\gamma_b\beta_b^2 c^2 k_{p0}^z}{\epsilon} x & E_y^a = -\frac{m\gamma_b\beta_b^2 c^2 k_{p0}^z}{\epsilon} y \end{cases}$$

$$k_{p0}^z = \text{const.}$$

Then the equations of motion reduce to:

$$\vec{X}_\perp'' + \left[\frac{1}{\gamma_b\beta_b ds} \frac{d(\gamma_b\beta_b)}{ds} \right] \vec{X}_\perp' + k_{p0}^z \vec{X}_\perp = -\frac{q}{m\gamma_b\beta_b^3 c^2} \frac{d\phi}{dX_\perp}$$

This situation is realized by a stationary (mass $\rightarrow \infty$), partially neutralizing uniform background of charges.

To see this we apply Poisson's equation to the applied field to calculate the neutralizing charge density:

$$\rho_a = \epsilon_0 \nabla \cdot \vec{E}^a = -2m\epsilon_0 \gamma_b \beta_b^2 c^2 k_{p0}^z = \text{const.}$$

- Unphysical model, but very commonly employed since it simply represents the average action of more physical focusing fields.
 - Demonstrate this later in some simple problems
- We will find that continuous focusing can provide reasonably good estimates for more realistic focusing models if k_{p0}^z is appropriately identified in terms of relevant factors.

In more physical situations one requires that ~~quasi-static~~ focusing fields in the aperture satisfy the vacuum Maxwell Equations in the beam region:

$\nabla \cdot \vec{E}^a = 0$	$\nabla \cdot \vec{B}^a = 0$
$\nabla \times \vec{E}^a = 0$	$\nabla \times \vec{B}^a = 0$

These equations constrain the 3D form of the vacuum fields resulting from spatially localized lenses. But the following situations can be exploited to optimize linear focusing in practical lattices: We will cover two common cases:

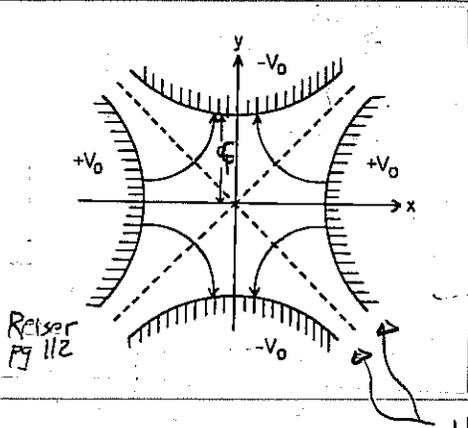
- 1) Alternating Gradient Quadrupoles
 - Electric Quadrupoles
 - Magnetic Quadrupoles
- 2) Magnetic Solenoids.

Alternating Gradient Quadrupole Focusing

A) Electric Quadrupoles: ($\vec{B}^q = 0$)

In the axial center of an electric quadrupole:

2D Fields



$$E_x^q = -\frac{2V_0}{r_p} \frac{x}{r_p} = -E'_z x$$

$$E_y^q = \frac{2V_0}{r_p} \frac{y}{r_p} = E'_z y$$

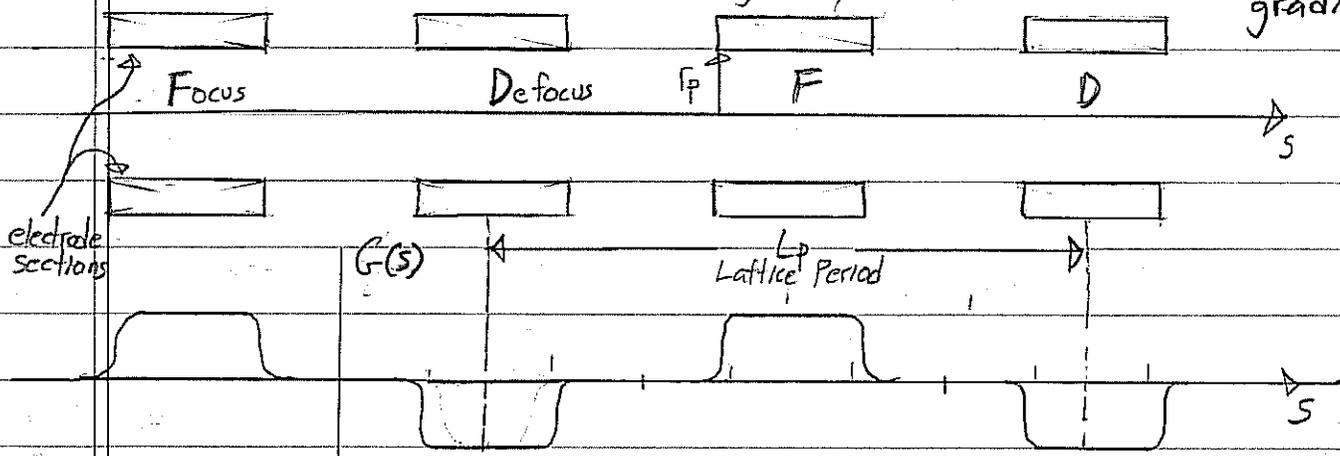
$$G = \frac{2V_0}{r_p^2} = -\frac{\partial E_x^q}{\partial x} = \frac{\partial E_y^q}{\partial y} = \text{Electric Gradient}$$

Hyperbolic Electrodes

$V_0 =$ pole voltage

$r_p =$ "pipe" aperture radius

For a lattice of finite length quadrupoles with alternating gradient



Equations of motion: Net focusing in both planes by alternating "Focus" and "Defocus" quadrupoles.

$$x'' + \left[\frac{1}{\gamma_b \beta_b} \frac{d(\gamma_b \beta_b)}{ds} \right] x' + \kappa_g(s) x = -\frac{q}{m \gamma_b^3 \beta_b^2 c^2} \frac{\partial \phi}{\partial x}$$

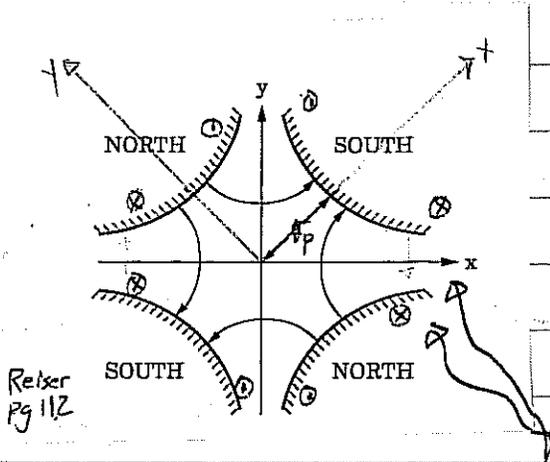
$$y'' + \left[\frac{1}{\gamma_b \beta_b} \frac{d(\gamma_b \beta_b)}{ds} \right] y' - \kappa_g(s) y = -\frac{q}{m \gamma_b^3 \beta_b^2 c^2} \frac{\partial \phi}{\partial y}$$

$$\kappa_g(s) \equiv \frac{q \cdot G(s)}{m \gamma_b^3 \beta_b^2 c^2} = \frac{G(s)}{\beta_b c [B_p]}$$

B) Magnetic Quadrupoles ($\vec{E}^a = 0$)

In the center of a magnetic quadrupole:

2D Fields



$$B_x^a = B_0 \frac{y}{r_p}$$

$$B_y^a = B_0 \frac{x}{r_p}$$

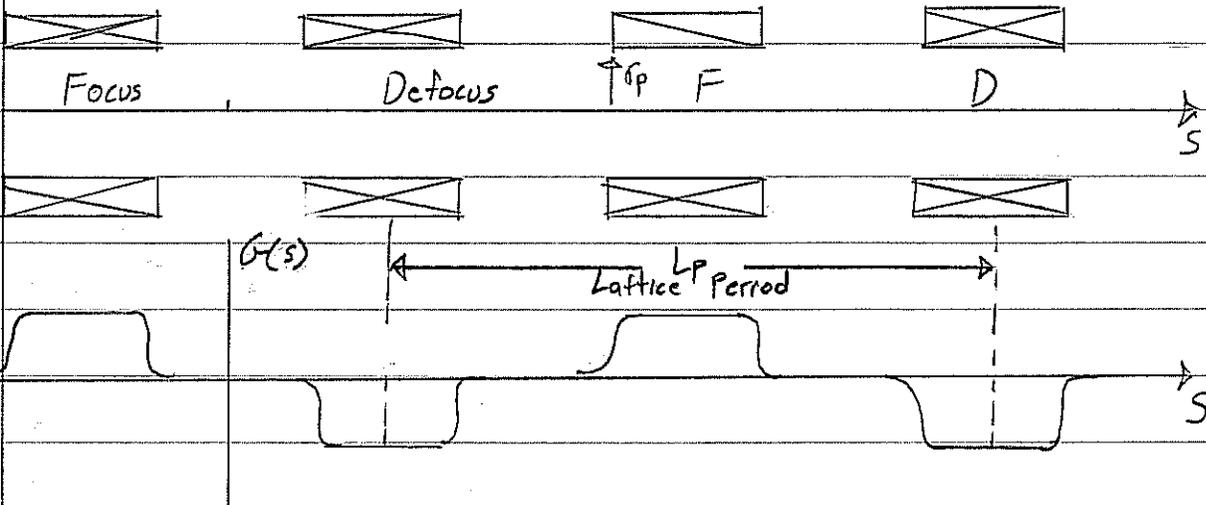
$$G \equiv \frac{B_0}{r_p} = \frac{\partial B_x^a}{\partial y} = \frac{\partial B_y^a}{\partial x} = \text{Magnetic Gradient}$$

$$B_0 = |\vec{B}_a|_{r=r_p} = \text{pole Field}$$

Hyperbolic Iron Poles.

$$r_p = \text{"pipe" aperture radius}$$

For a lattice of finite length quadrupoles with alternating gradient.



Equations of motion:

$$x'' + \left[\frac{1}{\gamma_b \beta_b} \frac{d(\gamma_b \beta_b)}{ds} \right] x' + K_g(s) x = \frac{-g}{m \gamma_b^3 \beta_b^2 c^2} \frac{\partial \phi}{\partial x}$$

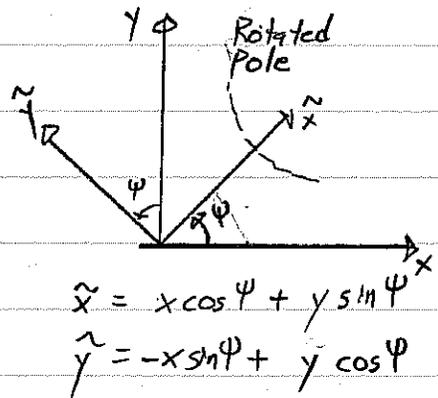
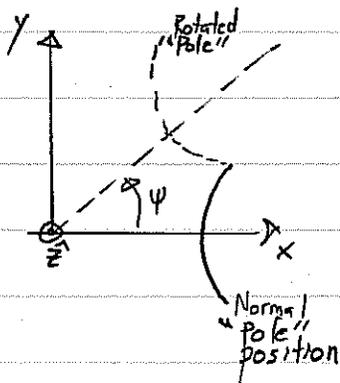
$$y'' + \left[\frac{1}{\gamma_b \beta_b} \frac{d(\gamma_b \beta_b)}{ds} \right] y' - K_g(s) x = \frac{-g}{m \gamma_b^3 \beta_b^2 c^2} \frac{\partial \phi}{\partial y}$$

$$K_g(s) \equiv \frac{g G(s)}{m \gamma_b \beta_b} = \frac{G(s)}{[B_p]}$$

* Equations are the same as for the electric case with diff. $K_g(s)$

Quadrupole Skew Couplings

Consider a quadrupole actively rotated through a positive angle Ψ about the z -axis.



$$\begin{aligned}\tilde{x} &= x \cos \Psi + y \sin \Psi \\ \tilde{y} &= -x \sin \Psi + y \cos \Psi\end{aligned}$$

Normal Orientation Fields:

Electric:

$$E_x^q = -Gx$$

$$E_y^q = Gy$$

Magnetic

$$B_x^q = Gy$$

$$B_y^q = Gx$$

Rotated Fields

Electric

$$E_x^q = E_{\tilde{x}}^q \cos \Psi - E_{\tilde{y}}^q \sin \Psi$$

$$E_y^q = E_{\tilde{x}}^q \sin \Psi + E_{\tilde{y}}^q \cos \Psi$$

$$E_{\tilde{x}}^q = -G\tilde{x} = -G(x \cos \Psi + y \sin \Psi)$$

$$E_{\tilde{y}}^q = G\tilde{y} = G(-x \sin \Psi + y \cos \Psi)$$

collect components:

$$E_x^q = -G(x \cos^2 \Psi + y \sin \Psi \cos \Psi) - G(-x \sin^2 \Psi + y \sin \Psi \cos \Psi)$$

$$= G(\sin^2 \Psi - \cos^2 \Psi)x - G/2 \sin \Psi \cos \Psi y$$

$$= -G \cos(2\Psi)x - G/2 \sin(2\Psi)y$$

$$E_y^q = G(x \sin \Psi \cos \Psi + y \sin^2 \Psi) + G(-x \sin \Psi \cos \Psi + y \cos^2 \Psi)$$

$$= -G \sin(2\Psi)x + G \cos(2\Psi)y$$

Magnetic

$$B_x^a = B_x^g \cos \Psi - B_y^g \sin \Psi$$

$$B_x^a = G \tilde{y} = G(-x \sin \Psi + y \cos \Psi)$$

$$B_y^a = B_x^g \sin \Psi + B_y^g \cos \Psi$$

$$B_y^a = G \tilde{x} = G(x \cos \Psi + y \sin \Psi)$$

collect components:

$$\begin{aligned} B_x^a &= G(-\sin \Psi \cos \Psi x + \cos^2 \Psi y) - G(\sin \Psi \cos \Psi x + y \sin^2 \Psi) \\ &= -G \sin(2\Psi) x + G \cos(2\Psi) y \end{aligned}$$

$$\begin{aligned} B_y^a &= G(-\sin^2 \Psi x + \sin \Psi \cos \Psi y) + G(\cos^2 \Psi x + \sin \Psi \cos \Psi y) \\ &= G \cos(2\Psi) x + G \sin(2\Psi) y \end{aligned}$$

Then the equations of motion become for both electric and magnetic focusing:

$$x'' + \frac{(\delta_b \beta_b)'}{(\delta_b \beta_b)} x' + R_g \cos(2\Psi) x + R_g \sin(2\Psi) y = \frac{-g}{m_b^3 \beta_b^2 c^2} \frac{\partial \phi}{\partial x}$$

$$y'' + \frac{(\delta_b \beta_b)'}{(\delta_b \beta_b)} y' - R_g \cos(2\Psi) y + R_g \sin(2\Psi) x = \frac{-g}{m_b^3 \beta_b^2 c^2} \frac{\partial \phi}{\partial y}$$

where as before

$$R_g = \begin{cases} \frac{G}{\beta_0 c [B_p]} & ; \text{Electric Focusing} \\ \frac{G}{[B_p]} & ; \text{Magnetic Focusing} \end{cases}$$

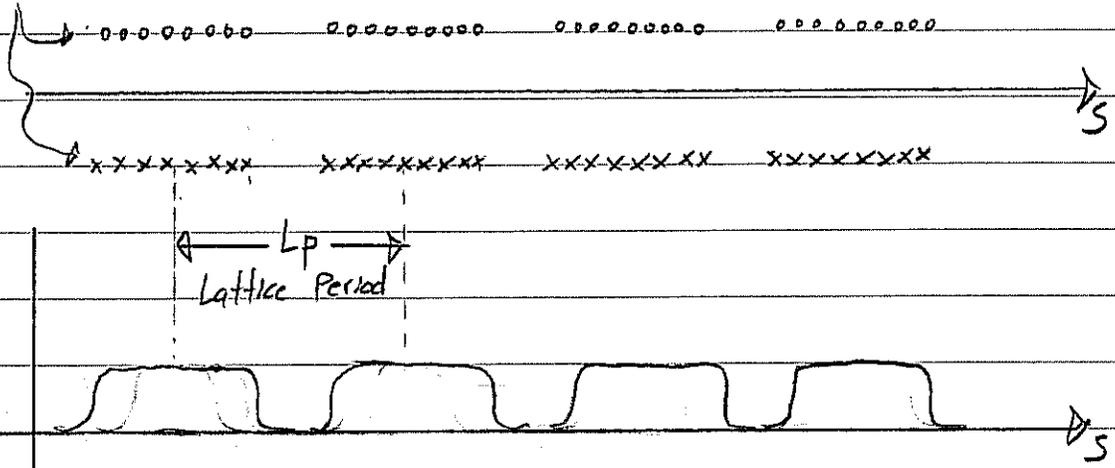
For $\Psi \neq 0, \pi, 2\pi, \dots$, applied field terms of R_g in the x -equation of motion depend on y and vice-versa for the y -equation of motion. Such "skew" coupling, while still being linear, considerably complicate the particle dynamics. Unless otherwise specified, quadrupoles are considered "normal" oriented. However, skew error terms or intentional skews can be important for some machines.

Solenoidal Focusing

$$\vec{E}^a = 0$$

$$\vec{B}^a = -\frac{1}{2} B_z^a(s) \vec{x}_\perp + \hat{z} B_z^a(s) + \text{Higher order Terms.}$$

Currents



Equations of motion:

$$x'' + \left[\frac{1}{\gamma_b \beta_b} \frac{d}{ds} (\gamma_b \beta_b) \right] x' = - \frac{\omega_c(s)}{2\gamma_b \beta_b c} y + \frac{\omega_c(s)}{\gamma_b \beta_b c} y' - \frac{g}{4m\gamma_b^3 \beta_b^2 c^2} \frac{\partial \phi}{\partial x}$$

$$y'' + \left[\frac{1}{\gamma_b \beta_b} \frac{d}{ds} (\gamma_b \beta_b) \right] y' = \frac{\omega_c(s)}{2\gamma_b \beta_b c} x - \frac{\omega_c(s)}{\gamma_b \beta_b c} x' - \frac{g}{4m\gamma_b^3 \beta_b^2 c^2} \frac{\partial \phi}{\partial y}$$

Here:

$\omega_c = \frac{g}{m} B_z^a(s)$ = cyclotron frequency
 of applied magnetic field
 depends on $B_z^a(s)$

These equations are linearly crosscoupled in the applied field terms

- x equation depends on $B_z^a y$, $B_z^a y'$
- y equation depends on $B_z^a x$, $B_z^a x'$

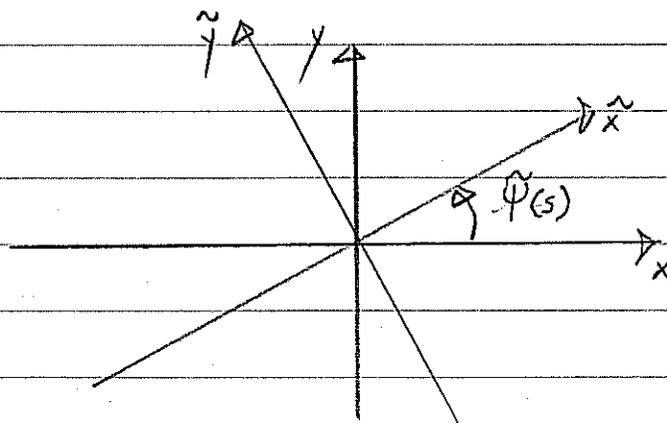
The linear cross-coupling in the applied field can be transformed away through a (s-varying) transformation to a rotating "Larmor" frame (see Appendix A). If the beam is axisymmetric

$$\frac{\partial \phi}{\partial \vec{x}_\perp} = \frac{\partial \phi}{\partial r} \frac{\partial r}{\partial \vec{x}_\perp} = \frac{\partial \phi}{\partial r} \frac{\vec{x}_\perp}{r}$$

then the equations of motion can be expressed in uncoupled form as (see Appendix A):

$$\begin{aligned} \tilde{x}'' + \frac{(\gamma_b \beta_b)'}{(\gamma_b \beta_b)} \tilde{x} + R(s) \tilde{x} &= \frac{-g}{m \gamma_b^3 \beta_b^2 c^2} \frac{\partial \phi}{\partial r} \frac{\tilde{x}}{r} \\ \tilde{y}'' + \frac{(\gamma_b \beta_b)'}{(\gamma_b \beta_b)} \tilde{y} + R(s) \tilde{y} &= \frac{-g}{m \gamma_b^3 \beta_b^2 c^2} \frac{\partial \phi}{\partial r} \frac{\tilde{y}}{r} \end{aligned}$$

Here:



Rotating Frame
variables with
~

$$R(s) = \frac{\omega_c^2(s)}{4 \gamma_b^2 \beta_b^2 c^2} \quad \text{Solenoid focusing strength.}$$

$$\begin{aligned} \tilde{x}(s) &= x \cos \tilde{\psi}(s) + y \sin \tilde{\psi}(s) \\ \tilde{y}(s) &= -x \sin \tilde{\psi}(s) + y \cos \tilde{\psi}(s) \end{aligned} \quad \tilde{\psi}(s) = -\frac{1}{2 \gamma_b \beta_b c} \int_{s_i}^s d\tilde{s} \omega_c(\tilde{s})$$

s_i : outside chain of magnets.

It has been shown that in certain reasonable approximations in a linear focusing channels including:

- 1) Continuous Focusing
- 2) Alternating Gradient Quadrupole Focusing
 - Electric
 - Magnetic
- 3) Solenoidal Focusing

that the transverse particle equations of motion can be cast in the form:

$$x'' + \left[\frac{1}{\gamma_b \beta_b} \frac{d}{ds} (\gamma_b \beta_b) \right] x' + K_x(s) x = \frac{-g}{m \gamma_b^3 \beta_b^2 c^2} \frac{\partial \phi}{\partial x}$$

$$y'' + \left[\frac{1}{\gamma_b \beta_b} \frac{d}{ds} (\gamma_b \beta_b) \right] y' + K_y(s) y = \frac{-g}{m \gamma_b^3 \beta_b^2 c^2} \frac{\partial \phi}{\partial y}$$

$K_x(s)$ = x - lattice focusing function

$K_y(s)$ = y - lattice focusing function

- Continuous Focusing: $K_x = K_y = k_{po}^2 = \text{const.}$
- Quadrupole focusing: $K_x(s) = -K_y(s) \equiv K_q(s)$
- Solenoidal focusing: $K_x(s) = K_y(s) \equiv K(s)$

Before studying space-charge effects, it is instructive to analyze these equations neglecting space-charge:

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial y} \rightarrow 0$$

and acceleration:

$$\frac{1}{\gamma_b \beta_b} \frac{d}{ds} (\gamma_b \beta_b) \rightarrow 0$$

In this simple case, the x - and y - equations have identical form:

$$x'' + K_x(s)x = 0$$

$$y'' + K_y(s)y = 0$$

Later we will show (KV Model) that linear space charge forces may also be included in K_x and K_y .

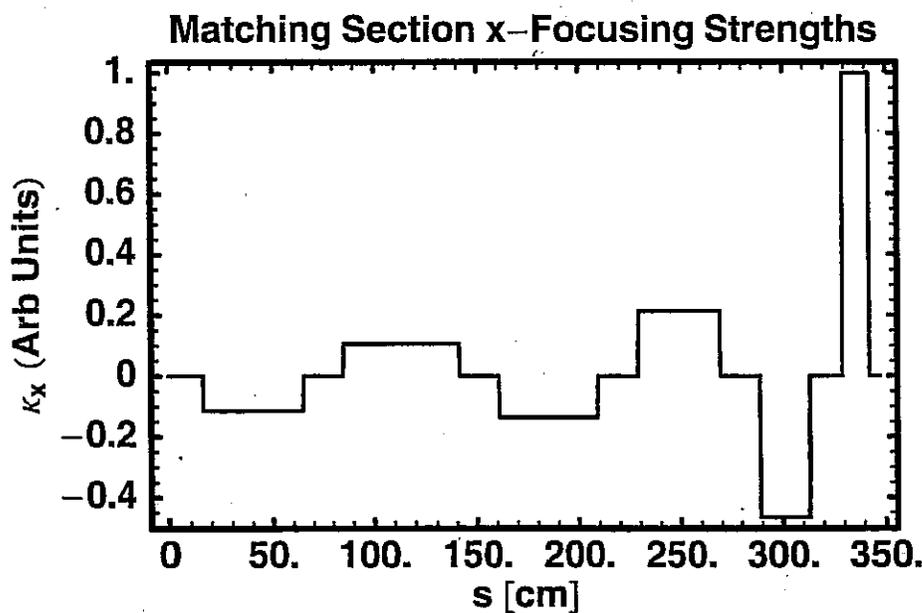
Usually, K_x and K_y will be periodic functions of s .

$$K_x(s+L_p) = K_x(s)$$

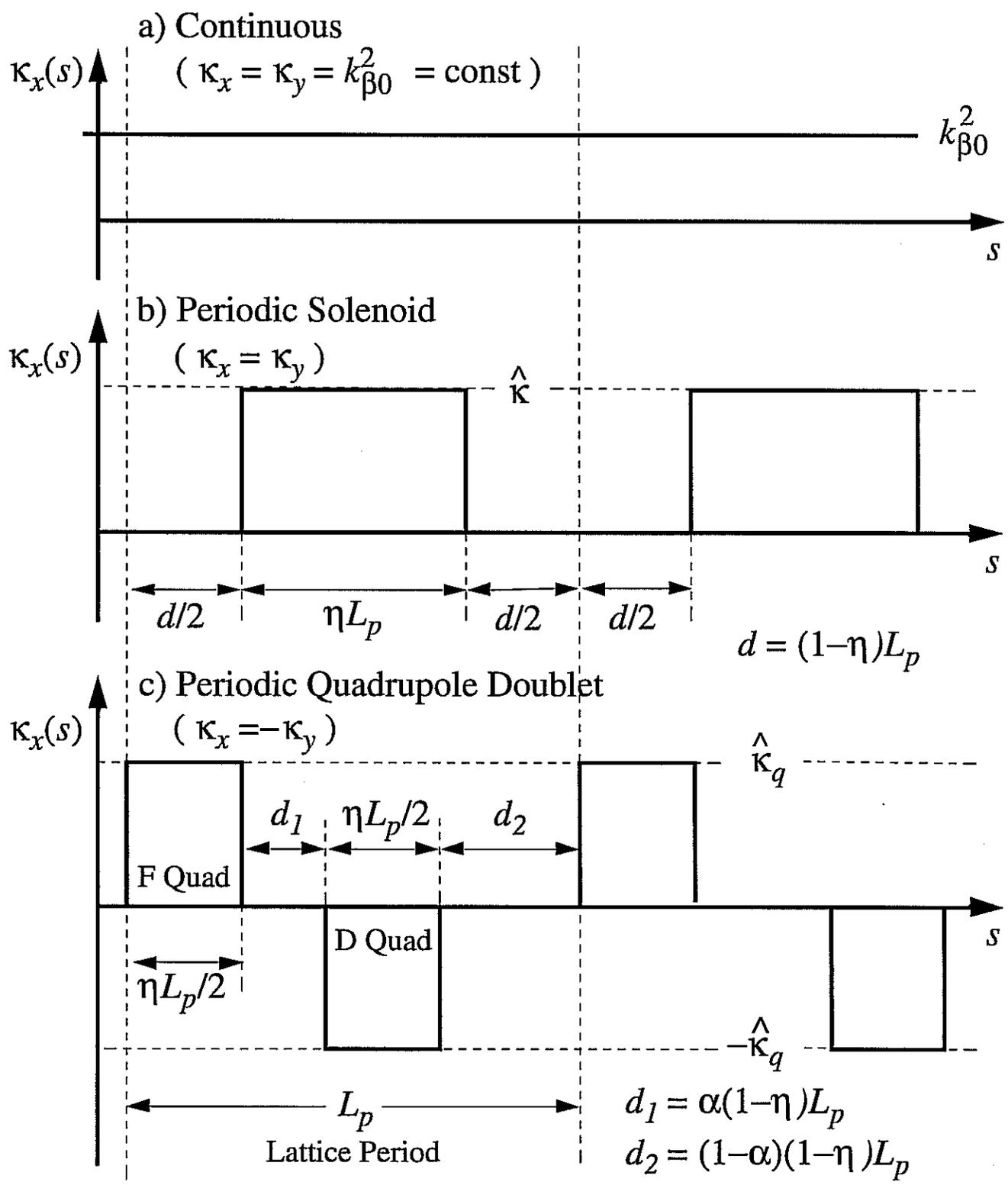
$$L_p = \text{lattice period}$$

$$K_y(s+L_p) = K_y(s)$$

But this need not be the case. Most of the analysis that follows is presented for the periodic case for simplicity in interpretation, but holds in general for matching sections and other non-periodic lattices where $K_x(s)$ and $K_y(s)$ may change form strongly



Piecewise Constant Linear Focusing Functions for Periodic Lattices



$$d_1 = \alpha(1-\eta)L_p$$

$$d_2 = (1-\alpha)(1-\eta)L_p$$

The equations presented in this section apply to a single particle moving in a beam.

In the remaining sections we will carry out analysis of the equations neglecting space-charge effects ($\phi \rightarrow 0$), as is conventional in the standard theory of low-intensity accelerators.

What we learn from this treatment will later aid analysis of high intensity beams ^{where space-charge cannot be neglected.} For

example, we will later see that the x and y particle equations of motion can be applied to analyze the evolution of the beam in the absence of image charges:

$$\begin{array}{ll}
 x \rightarrow & x_c = \langle x \rangle \quad x\text{-Centroid} \\
 y \rightarrow & y_c = \langle y \rangle \quad y\text{-Centroid}
 \end{array}$$

By making appropriate variable substitutions, much of the analysis that follows can be applied to intense beams.

§2 Appendix A: Transformation of Solenoidal Focused Particle A/V
Equations of Motion to Uncoupled Form

Solenoid Equations of motion:

$$x'' + \left[\frac{1}{\gamma_b \beta_b} \frac{d}{ds} (\gamma_b \beta_b) \right] x' = \frac{\omega_c'(s)}{2\gamma_b \beta_b c} y + \frac{\omega_c(s)}{\gamma_b \beta_b c} y' - \frac{q}{m\gamma_b^3 \beta_b^2 c^2} \frac{\partial \phi}{\partial x}$$

$$y'' + \left[\frac{1}{\gamma_b \beta_b} \frac{d}{ds} (\gamma_b \beta_b) \right] y' = -\frac{\omega_c'(s)}{2\gamma_b \beta_b c} x - \frac{\omega_c(s)}{\gamma_b \beta_b c} x' - \frac{q}{m\gamma_b^3 \beta_b^2 c^2} \frac{\partial \phi}{\partial y}$$

To simplify algebra introduce the complex coordinate

$$z = x + iy$$

Then the two equations can be expressed as a single complex equation:

$$z'' + \left[\frac{1}{\gamma_b \beta_b} \frac{d}{ds} (\gamma_b \beta_b) \right] z' + \frac{i\omega_c'}{2\gamma_b \beta_b c} z + \frac{i\omega_c}{\gamma_b \beta_b c} z' = \frac{-q}{m\gamma_b^3 \beta_b^2 c^2} \left(\frac{\partial \phi}{\partial x} + i \frac{\partial \phi}{\partial y} \right)$$

For axisymmetric potential ϕ :

$$\frac{\partial \phi}{\partial x} + i \frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial r} \frac{z}{r} \quad r = \sqrt{x^2 + y^2}$$

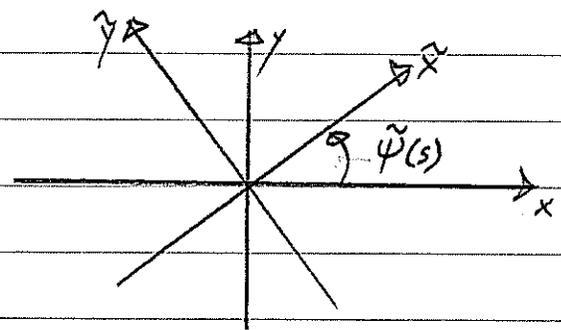
and the equation of motion becomes:

$$z'' + \frac{(\gamma_b \beta_b)'}{(\gamma_b \beta_b)} z' + \frac{i\omega_c'}{2\gamma_b \beta_b c} z + \frac{i\omega_c}{\gamma_b \beta_b c} z' = \frac{-q}{m\gamma_b^3 \beta_b^2 c^2} \frac{\partial \phi}{\partial r} \frac{z}{r}$$

$$z'' + \frac{(\gamma_b \beta_b)'}{(\gamma_b \beta_b)} z' + \frac{i\omega_c'}{2\gamma_b \beta_b c} z + \frac{i\omega_c}{\gamma_b \beta_b c} z' = \frac{-q}{m\gamma_b^3 \beta_b^2 c^2} \frac{\partial \phi}{\partial r} \frac{z}{r}$$

Following Wredemann, Vol. II pg 82, S.M. Lund AZ/
 Introduce the complex variable \tilde{z} and a (real) phase function $\tilde{\psi}(s)$ by:

$$\tilde{z} = z e^{-i\tilde{\psi}(s)} \equiv \tilde{x} + i\tilde{y}$$



Note:
 $|\tilde{z}| = r$

Then

$$\begin{aligned} z &= \tilde{z} e^{i\tilde{\psi}} \\ \dot{z} &= (\tilde{z}' + i\tilde{\psi}'\tilde{z}) e^{i\tilde{\psi}} \\ \ddot{z} &= (\tilde{z}'' + 2i\tilde{\psi}'\tilde{z}' + i\tilde{\psi}''\tilde{z} - \tilde{\psi}'^2\tilde{z}) e^{i\tilde{\psi}} \end{aligned}$$

and the complex equations of motion become:

$$\begin{aligned} \tilde{z}'' + \left[2i\tilde{\psi}' + i\omega_c + \frac{(\gamma_b \beta_b)'}{\gamma_b \beta_b c} \right] \tilde{z}' + \left[i\tilde{\psi}'' - \tilde{\psi}'^2 - \omega_c \tilde{\psi}' + i\omega_c' + i\tilde{\psi}' \frac{(\gamma_b \beta_b)'}{\gamma_b \beta_b c} \right] \tilde{z} \\ = -\frac{g}{m\gamma_b^3 \beta_b^2 c^2} \frac{d\phi}{ds} \frac{\tilde{z}}{r} \end{aligned}$$

Choose $\tilde{\psi}(s)$ (free to make this choice ... note that it eliminates the imaginary terms in []s.)

$$\tilde{\psi}' = -\frac{\omega_c}{2\gamma_b \beta_b c}$$

Then $\tilde{\psi}'' = -\frac{\omega_c'}{2\gamma_b \beta_b c} + \frac{\omega_c}{2c} \frac{(\gamma_b \beta_b)'}{(\gamma_b \beta_b)^2}$ and the equation to reduces to

$$\tilde{z}'' + \frac{(\gamma_b \beta_b)'}{(\gamma_b \beta_b)} \tilde{z}' + \frac{\omega_c^2}{4\gamma_b^2 \beta_b^2 c^2} \tilde{z} = -\frac{g}{m\gamma_b^3 \beta_b^2 c^2} \frac{d\phi}{ds} \frac{\tilde{z}}{r}$$

Or using $\tilde{y} = \hat{x} + i\tilde{y}$, the equations are now decoupled in the \tilde{x}, \tilde{y} variables!

$$\tilde{x}'' + \frac{(\gamma_b \beta_b)'}{(\gamma_b \beta_b)} \tilde{x} + K_s(s) \tilde{x} = -\frac{g}{m \gamma_b^3 \beta_b^2 c^2} \frac{d\phi}{ds} \frac{\partial}{\partial r} \tilde{x}$$

$$\tilde{y}'' + \frac{(\gamma_b \beta_b)'}{(\gamma_b \beta_b)} \tilde{y} + K_s(s) \tilde{y} = -\frac{g}{m \gamma_b^3 \beta_b^2 c^2} \frac{d\phi}{ds} \frac{\partial}{\partial r} \tilde{y}$$

Uncoupled \Rightarrow
form same "
as for
continuous
or quadrupole
focusing.

Solenoidal
focusing
force

$$K_s(s) \equiv \frac{\omega_c^2(s)}{4\gamma_b^2 \beta_b^2 c^2}$$

$$\omega_c(s) = \frac{g B_z(s)}{m}$$

The transformation can be effected by integrating the equation for $\tilde{\psi}'$

$$\tilde{\psi}(s) = -\frac{1}{2\gamma_b \beta_b c} \int_{s_i}^s d\tilde{s} \omega_c(\tilde{s})$$

where s_i some value of s where the solenoidal field $B_z(s)$ is zero, (i.e., outside the lattice of magnets). Then the transformation is given by:

$$\tilde{x} = \operatorname{Re} \left[(x + iy) e^{-i\tilde{\psi}(s)} \right] = x \cos \tilde{\psi}(s) + y \sin \tilde{\psi}(s)$$

$$\tilde{y} = \operatorname{Im} \left[(x + iy) e^{-i\tilde{\psi}(s)} \right] = -x \sin \tilde{\psi}(s) + y \cos \tilde{\psi}(s)$$

Since $\tilde{\psi}' = -\frac{1}{2} \left(\frac{\omega_c}{\gamma_b \beta_b c} \right)$, note that the \tilde{x}, \tilde{y} frame is rotating at $1/2$ the local (s -varying) cyclotron frequency. This rotating frame is referred to as the "Larmor" frame. For a

uniform solenoid with $B_z(s) = \text{const}$,

$\tilde{\psi}' = -\frac{1}{2} \left(\frac{\omega c}{\gamma \beta_0 \beta_0 c} \right) = \text{const}$, and the Larmor frame is uniformly rotating as is well-known.

Note that at the initial condition with $B_z = 0$,

$$\tilde{x}(s=s_i) = x(s=s_i) = x_i$$

$$\tilde{x}'(s=s_i) = x'(s=s_i) = x'_i$$

$$\tilde{y}(s=s_i) = y(s=s_i) = y_i$$

$$\tilde{y}'(s=s_i) = y'(s=s_i) = y'_i$$

The rotation matrix of the transformation can be explicitly worked out:

$$\tilde{x} = x \cos \tilde{\psi} + y \sin \tilde{\psi}$$

$$\tilde{x}' = -x \tilde{\psi}' \sin \tilde{\psi} + x' \cos \tilde{\psi} + y \tilde{\psi}' \cos \tilde{\psi} + y' \sin \tilde{\psi}$$

$$\tilde{y} = -x \sin \tilde{\psi} + y \cos \tilde{\psi}$$

$$\tilde{y}' = -x \tilde{\psi}' \cos \tilde{\psi} - x' \sin \tilde{\psi} - y \tilde{\psi}' \sin \tilde{\psi} + y' \cos \tilde{\psi}$$

collecting terms and using $\tilde{\psi}' = -\frac{1}{2} \frac{\omega c}{\gamma \beta_0 \beta_0 c}$;

$$\begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix} = \begin{pmatrix} \cos \tilde{\psi} & 0 & \sin \tilde{\psi} & 0 \\ \frac{\omega c}{2\gamma \beta_0 \beta_0 c} \sin \tilde{\psi} & \cos \tilde{\psi} & -\frac{\omega c}{2\gamma \beta_0 \beta_0 c} \cos \tilde{\psi} & \sin \tilde{\psi} \\ -\sin \tilde{\psi} & 0 & \cos \tilde{\psi} & 0 \\ \frac{\omega c}{2\gamma \beta_0 \beta_0 c} \cos \tilde{\psi} & -\sin \tilde{\psi} & \frac{\omega c}{2\gamma \beta_0 \beta_0 c} \sin \tilde{\psi} & \cos \tilde{\psi} \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{x}' \\ \tilde{y} \\ \tilde{y}' \end{pmatrix}$$

§3 Description of Applied Focusing Fields

Applied fields for focusing, bending, and acceleration enter the equations of motion via

\vec{E}^a : Applied Electric Field

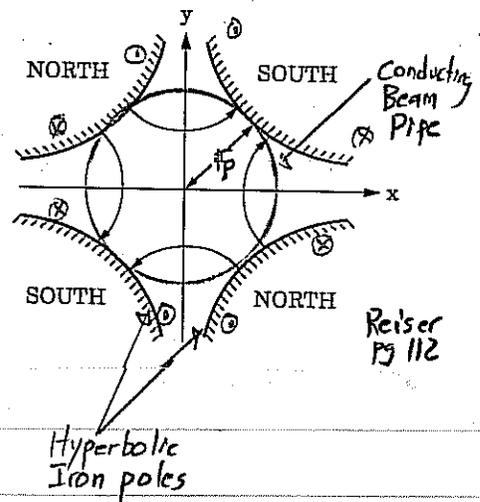
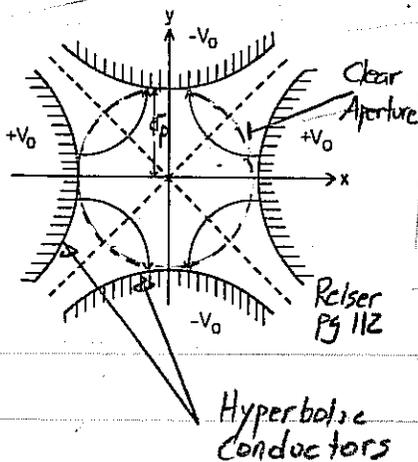
\vec{B}^a : Applied Magnetic Field

Generally, these fields are produced by sources outside an aperture of radius r_p . For example, for electric and magnetic quadrupoles:

Electric Quadrupole

Magnetic Quadrupole

Transverse
 Cross-
 Sections



The fields within the aperture of these and similar elements satisfy the vacuum Maxwell Equations:

$\nabla \cdot \vec{E}^a = 0$ $\nabla \times \vec{E}^a = -\frac{\partial \vec{B}^a}{\partial t}$	$\nabla \cdot \vec{B}^a = 0$ $\nabla \times \vec{B}^a = \frac{1}{c^2} \frac{\partial \vec{E}^a}{\partial t}$
---	--

In many cases of interest, the fields are static or a quasi-static approximation can be employed and the $\partial/\partial t$ terms neglected. Then the equations for \vec{E}^q and \vec{B}^q within the aperture are of the same form with:

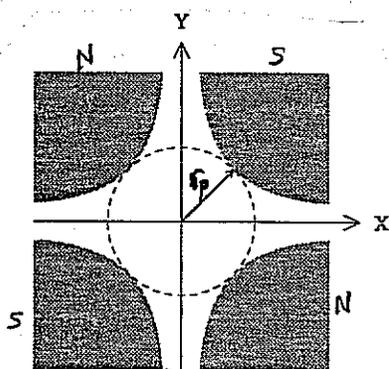
$\nabla \cdot \vec{E}^q = 0$	$\nabla \cdot \vec{B}^q = 0$	Static Vacuum Maxwell Equations
$\nabla \times \vec{E}^q = 0$	$\nabla \times \vec{B}^q = 0$	

In general, elements are tuned to limit the strength of nonlinear field terms, so the beam experiences primarily linear applied fields. We will find later that this allows better preservation of beam quality. However, this removal of nonlinear field terms cannot be done absolutely in 3D geometries. There will always be some nonlinear fields in finite 3D structures as a result of the properties of the vacuum Maxwell equations. Also, even in lower-dimensional idealizations of real structures, deviations from ideal symmetry will result from practical concerns and such deviations will produce further nonlinear field terms. As an example of this, 2D iron magnet with infinite pole extent must be truncated leading to nonlinear focusing terms even in 2D:

Cross-sections of Quadrupole Magnets

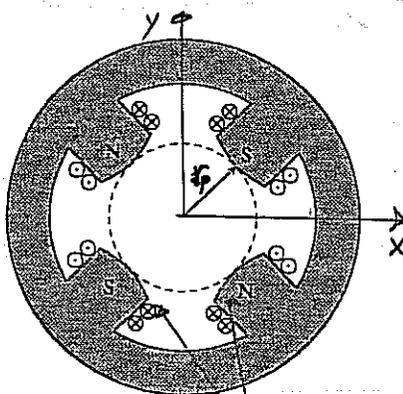
(Conte and MacKay, Intro. to the Physics of Particle Accelerators, World Scientific, 1991)
Pg 59.

Ideal



Hyperbolic iron poles : 2D Field
Ideal Quadrupole

Practical



Truncated iron poles : 2D Field
Ideal Quadrupole
+ Error Terms

Magnetic optics and accelerating structures are both extensive topics and it is not possible to cover them in this intro. (survey) class. In the following we will illustrate some simple methods of field error characterization of magnetic optics with transverse field components (dipoles, quadrupoles, sextupoles, ...)

This treatment is not comprehensive.

Consider magnetic optics with:

$$\vec{E}^a = 0 \quad \text{and} \quad \vec{B}^a \simeq \vec{B}_\perp^a$$

Electric optics can be treated analogously since the static vacuum Maxwell Equations are the same form for \vec{E}^a and \vec{B}^a .

Magnetic Forces

$$\vec{F}_\perp^a \approx g \beta_{bc} \cdot \hat{z} \times \vec{B}_\perp^a$$

$$\vec{B}_\perp^a = \hat{x} B_x^a + \hat{y} B_y^a \quad \perp \text{ Field}$$

$$\begin{aligned} F_x^a &\approx -g \beta_{bc} \cdot B_y^a \\ F_y^a &\approx g \beta_{bc} \cdot B_x^a \end{aligned}$$

Field components can be Taylor-series expanded about the element center $\vec{x}_i = 0$

$$B_x^a = B_x^a(0) + \frac{\partial B_x^a(0)}{\partial y} y + \frac{\partial B_x^a(0)}{\partial x} x$$

Bend Normal Linear Focus Skew Linear Focus

$$\left. \begin{aligned} &+ \frac{1}{2} \frac{\partial^2 B_x^a(0)}{\partial x^2} x^2 + \frac{\partial^2 B_x^a(0)}{\partial x \partial y} xy + \frac{1}{2} \frac{\partial^2 B_x^a(0)}{\partial y^2} y^2 \\ &+ \dots \end{aligned} \right\} \text{Nonlinear Terms}$$

$$B_y^a = B_y^a(0) + \frac{\partial B_y^a(0)}{\partial x} x + \frac{\partial B_y^a(0)}{\partial y} y$$

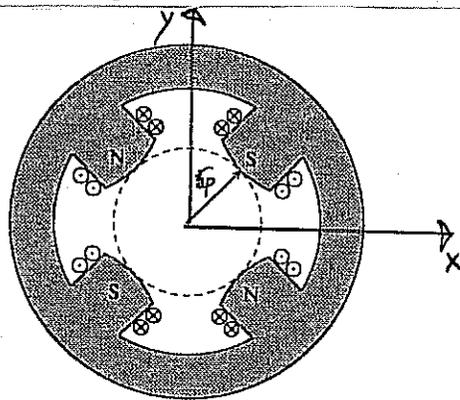
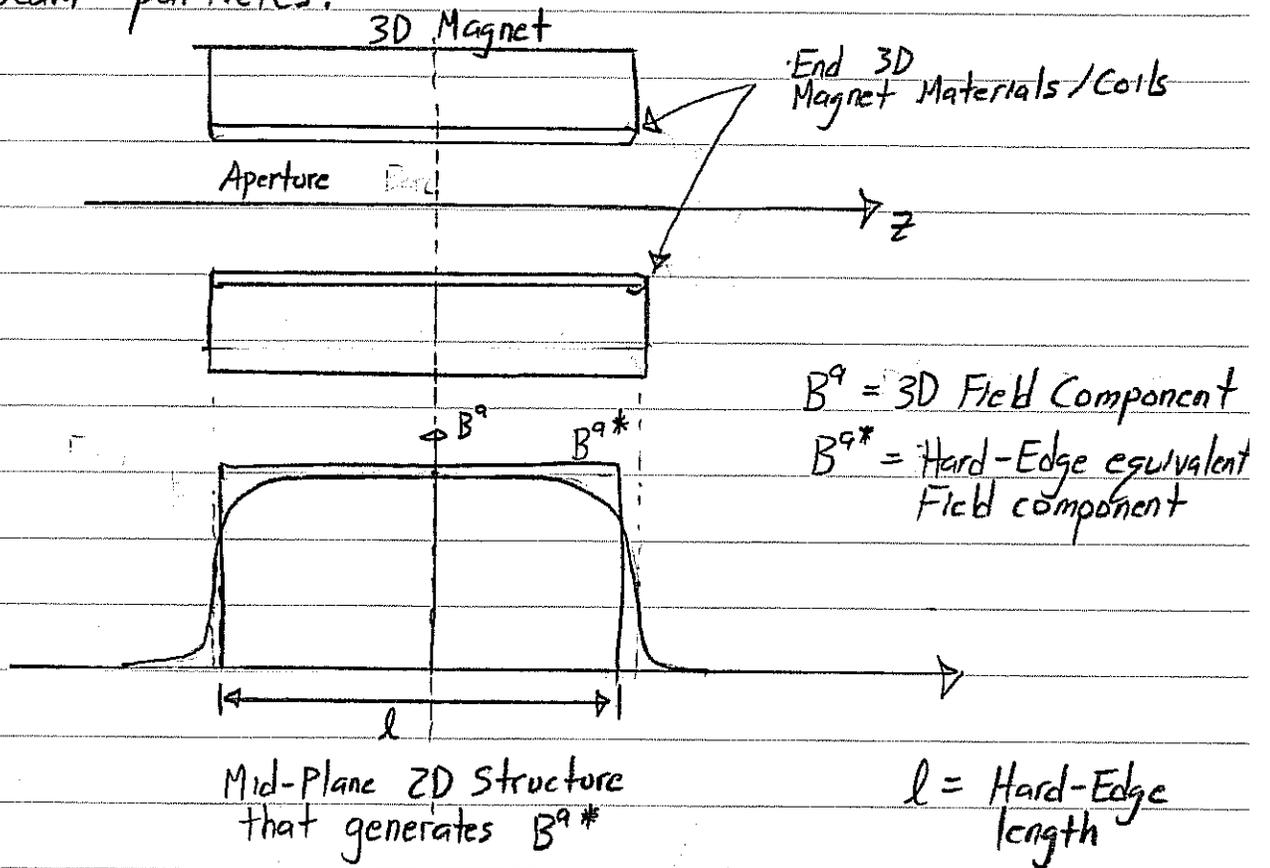
Bend Normal Linear Focus Skew Linear Focus

$$\left. \begin{aligned} &+ \frac{1}{2} \frac{\partial^2 B_y^a(0)}{\partial x^2} x^2 + \frac{\partial^2 B_y^a(0)}{\partial x \partial y} xy + \frac{1}{2} \frac{\partial^2 B_y^a(0)}{\partial y^2} y^2 \\ &+ \dots \end{aligned} \right\} \text{Nonlinear Terms}$$

Sources of Undesired Applied Field Components:

- 1/ Systematic errors or nonideal geometry associated with the intrinsic structure of the magnet.
- 2/ Random construction errors in individual magnets.
- 3/ Alignment of magnets giving field projections in unwanted directions
- 4/ Excitation errors (currents in coils not correct values and/or balanced).

Real 3D magnets can often be modeled for sufficient accuracy by 2D hard-edge equivalent magnets that give same impulse to the beam particles.



Many prescriptions exist for calculating l and optimal hard-edge models for various purposes. Here we follow, Lund and Bukh, PRSTAB 7 204801 (2004) Appendix C

A reasonable prescription to calculate l for a physical quadrupole magnet with a linear field gradient G is:

$$G(z) \equiv \left. \frac{\partial B_x^a}{\partial y} \right|_{x=y=0}$$

$$l G(z=0) = \int_{-\infty}^{\infty} dz G(z)$$

l = "effective"
magnet length

In many cases, it is sufficient to characterize the field errors in 2D hard-edge equivalent as:

$$B_x(x, y) = \frac{1}{l} \int_{-\infty}^{\infty} dz B_x^a(x, y, z)$$

$$B_y(x, y) = \frac{1}{l} \int_{-\infty}^{\infty} dz B_y^a(x, y, z)$$

\uparrow 2D Effective Fields \uparrow 3D Fields

Operating on the vacuum Maxwell equations with $\int_{-\infty}^{\infty} \frac{dz}{l}$ yields the (exact) 2D field equations:

$$\frac{\partial B_x(x, y)}{\partial x} = -\frac{\partial B_y(x, y)}{\partial y}$$

$$\frac{\partial B_x(x, y)}{\partial y} = \frac{\partial B_y(x, y)}{\partial x}$$

These equations can be recognized as the Cauchy-Riemann conditions for a complex field variable

$$\tilde{B} = B_y + i B_x \quad \text{Here, } i = \sqrt{-1}$$

to be an analytical function of the complex variable

$$\tilde{z} = x + i y \quad i = \sqrt{-1}$$

It follows that $B_{\sim}(z)$ can be analyzed using the full power of the highly developed theory of analytical functions of a complex variable.

Expand $B_{\sim}(z)$ as a Laurent series within the vacuum aperture as:

$$B_{\sim}(z) = B_y + i B_x = \sum_{n=1}^{\infty} B_n \left(\frac{z}{r_p} \right)^{n-1}$$

$B_n = \text{const}$ complex constants (multipole coefficients) giving the structure of the field.

$n =$ multipole index

$r_p =$ aperture "pipe" radius

The multipole coefficients can be resolved as:

$$B_n = b_n + i a_n \quad a_n, b_n \text{ real constants}$$

$b_n \Rightarrow$ "normal" multipoles

$a_n \Rightarrow$ "skew" multipoles (rotated)

Index	Name	Normal Field
$n=1$	Dipole	$B_x = 0, \quad B_y = b_1$
$n=2$	Quadrupole	$B_x = b_2 y/r_p, \quad B_y = b_2 x/r_p$
$n=3$	Sextupole	$B_x = -b_3 2xy/r_p^2, \quad B_y = b_3 (x^2 - y^2)/r_p^2$
$n=4$	Octupole	
\vdots	\vdots	\vdots

Higher order multipole field components leading to nonlinear focusing forces decrease rapidly within the aperture. To see this use a polar representation of \underline{z} and \underline{B}_n :

$$\underline{z} = x + iy = r e^{i\theta}$$

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \text{ArcTan}[y, x]$$

similarly:

$$\underline{B}_n = |B_n| e^{i\psi}$$

Thus the n -th order multipole term scales as

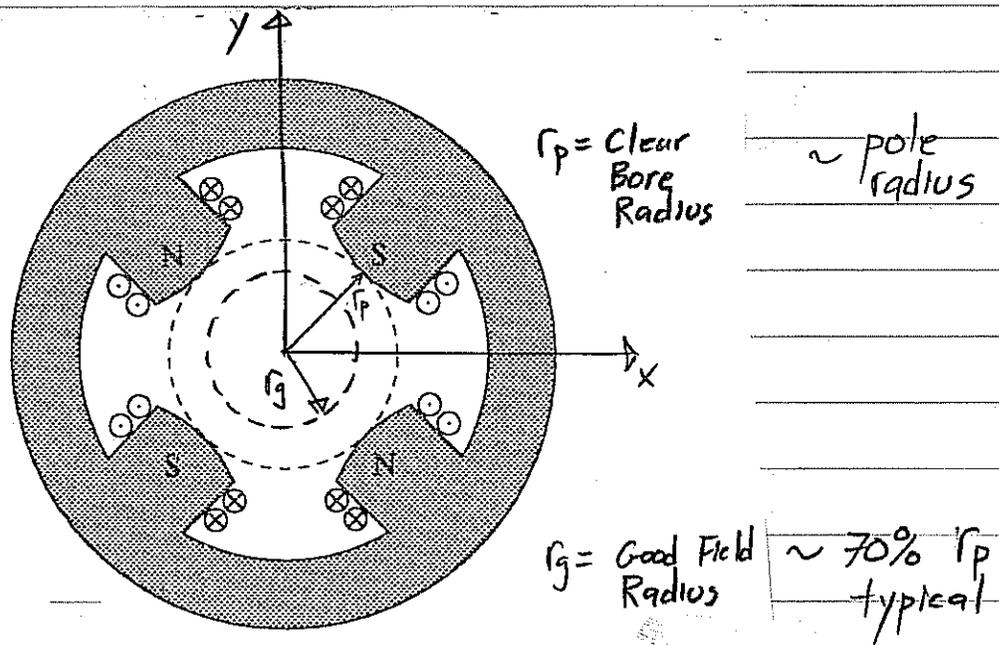
$$\underline{B}_n \left(\underline{z} / r_p \right)^{n-1} = |B_n| \left(\frac{r}{r_p} \right)^{n-1} e^{i[(n-1)\theta + \psi]}$$

- Unless the coefficient $|B_n|$ is very large, high-order terms will become small rapidly as (r/r_p) decreases

Better linear field quality magnets can be made simply by making the clear bore r_p larger or alternatively smaller beam cross-sections can be employed.

- Larger bore magnets cost more.
So design becomes trade-off between cost and performance.

Generally, a practical magnet design will have a "good-field" radius that maximum field errors are specified on. In a good magnet design this can be 70% or more of the clear bore to the beginning of magnet material structures.



Beam particle orbits are designed to remain within radius r_g .

Field error statements are readily generalized to 3D since

$$\begin{aligned} \nabla \cdot \vec{B}^g &= 0 \\ \nabla \times \vec{B}^g &= 0 \end{aligned} \quad \Rightarrow \quad \nabla^2 \vec{B}^g = 0$$

each component of \vec{B}^g satisfies a Laplace equation within the vacuum aperture. Therefore, field errors can only decrease when moving deeper within a source free region of 3D space.

Example 2D Nonideal Magnets from Permanent Magnet blocks

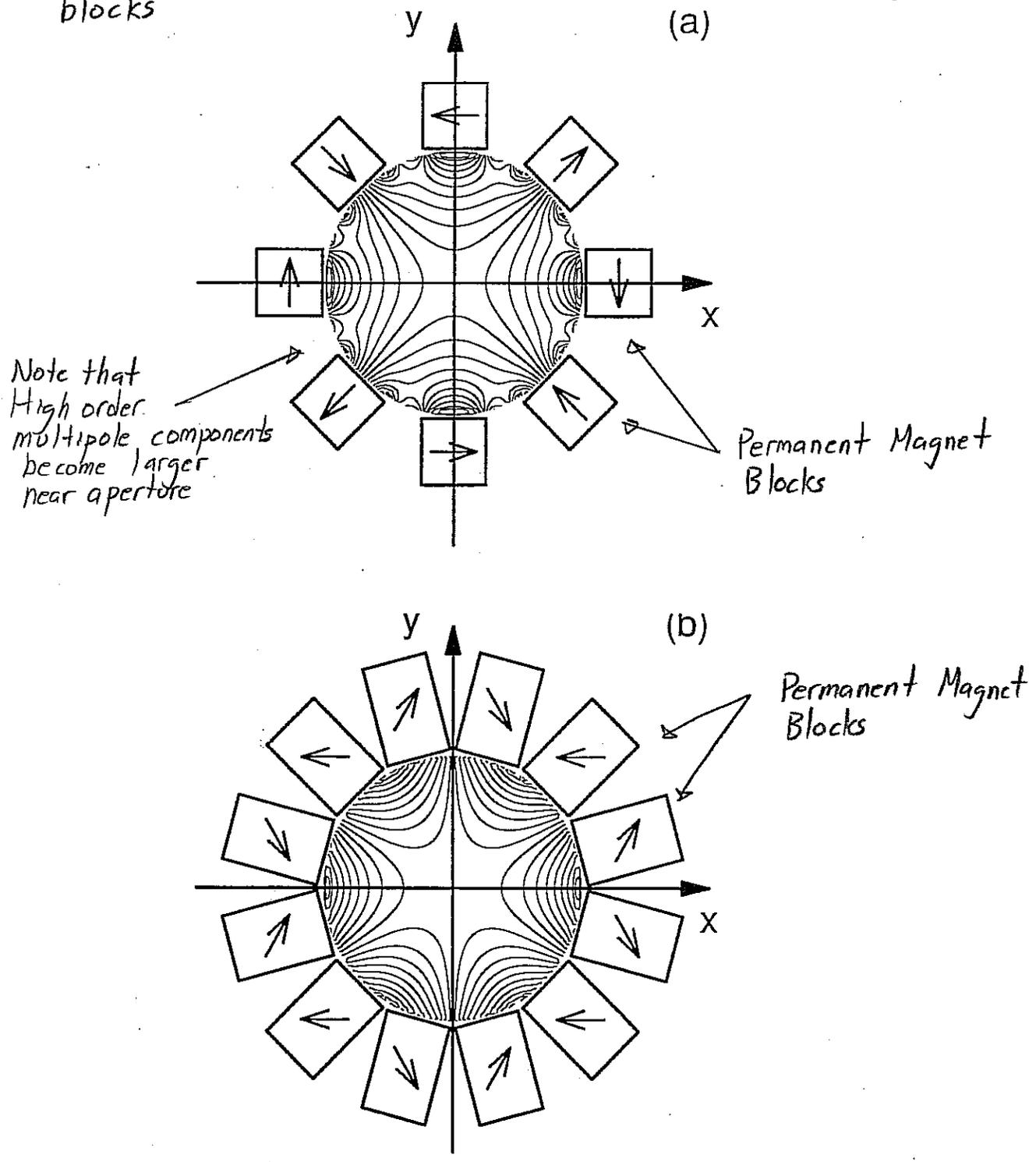


Fig. E1. Cross-sections of a Halbach array quadrupole magnet formed from $M = 8$ square-block magnets (a), and a sextupole magnet formed from $M = 12$ rectangular-block magnets (b).

§4 Transverse Particle Equations
of Motion with Nonlinear
Applied Fields

In §1 we showed that

$$\begin{aligned}
 x'' + \frac{(\gamma_b \beta_b)'}{(\gamma_b \beta_b)} x' &= \frac{q}{m \gamma_b \beta_b c^2} E_x^a - \frac{q}{m \gamma_b \beta_b c} B_y^a + \frac{q}{m \gamma_b \beta_b c} B_z^a y' \\
 &\quad - \frac{q}{m \gamma_b \beta_b c^2} \frac{\partial \phi}{\partial x} \\
 y'' + \frac{(\gamma_b \beta_b)'}{(\gamma_b \beta_b)} y' &= \frac{q}{m \gamma_b \beta_b c^2} E_y^a + \frac{q}{m \gamma_b \beta_b c} B_x^a - \frac{q}{m \gamma_b \beta_b c} B_z^a x' \\
 &\quad - \frac{q}{m \gamma_b \beta_b c^2} \frac{\partial \phi}{\partial y}
 \end{aligned}$$

for linear applied fields (§2) these equations can be expressed as:

$$\begin{aligned}
 x'' + \frac{(\gamma_b \beta_b)'}{(\gamma_b \beta_b)} x' + r_x x &= -\frac{q}{m \gamma_b \beta_b c^2} \frac{\partial \phi}{\partial x} \\
 y'' + \frac{(\gamma_b \beta_b)'}{(\gamma_b \beta_b)} y' + r_y y &= -\frac{q}{m \gamma_b \beta_b c^2} \frac{\partial \phi}{\partial y}
 \end{aligned}$$

where

$$r_x(s), r_y(s)$$

describe the linear applied focusing forces and the equations are implicitly treated in the Larmor frame when $B_z^a \neq 0$.

Lattice designs try to minimize the effect of nonlinear applied fields. However, the 3D Maxwell

equations predict that there will always be some finite nonlinear applied field terms in 3D.

Moreover, design idealizations and fabrication and material errors will generate further nonlinear field components in practical devices.

Therefore, nonlinear field effects must be added back into the model when it is necessary to analyze their effects.

There are several approaches to carrying this out:

Approach 1 (Explicit 3D)

Simplist - Just employ the explicit 3D applied field components \vec{E}_a , \vec{B}_a in the full equations of motion and avoid using R_x, R_y .

Comments:

- Most easy to apply without unit confusion in simulations where many effects are included.
 - Simplify comparisons to experiments where details matter?
- The accelerating field E_z^a associated with longitudinal electric fields should also be included to calculate $\gamma_b(s)$ and $\beta_b(s)$ for accurate transverse equations of motion.
 - \perp and \parallel dynamics cannot be fully decoupled for high accuracy?

Approach 2 (Perturbations about linear)

Exploit the linearity of the Maxwell's equation to take

$$\begin{array}{l} \vec{E}_\perp^q = \vec{E}_\perp^q \Big|_{\text{Linear Focus}} + \delta \vec{E}_\perp^q \\ \vec{B}^q = \vec{B}^q \Big|_{\text{Linear Focus}} + \delta \vec{B}^q \end{array}$$

where $\vec{E}_\perp^q \Big|_{\text{Linear Focus}}$ and $\vec{B}^q \Big|_{\text{Linear Focus}}$ are the linear field components included in k_x, k_y and then express the equations of motion as:

$$\begin{aligned} x'' + \frac{(\gamma_b \beta_b)'}{(\gamma_b \beta_b)} x' + k_x x &= \frac{q}{m \gamma_b^3 \beta_b^2 c^2} \delta E_x^q - \frac{q}{m \gamma_b \beta_b c} \delta B_y^q + \frac{q}{m \gamma_b \beta_b c} \delta B_z^q y' \\ &\quad - \frac{q}{m \gamma_b^3 \beta_b^2 c^2} \frac{\partial \phi}{\partial x} \\ y'' + \frac{(\gamma_b \beta_b)'}{(\gamma_b \beta_b)} y' + k_y y &= \frac{q}{m \gamma_b \beta_b c} \delta E_y^q + \frac{q}{m \gamma_b \beta_b c} \delta B_x^q - \frac{q}{m \gamma_b \beta_b c} \delta B_z^q x' \\ &\quad - \frac{q}{m \gamma_b^3 \beta_b^2 c^2} \frac{\partial \phi}{\partial y} \end{aligned}$$

Comments

- Approach 1 is best suited to non-solenoidal focusing because for solenoidal focusing the form is only valid for $\phi = \phi(r)$ (Axisymmetric beam) and the $\delta \vec{E}_\perp$ and $\delta \vec{B}$ field components would need to be projected into the Larmor frame. This would be more complicated than Approach 1.

- The field components $\delta \vec{E}_\perp^a$ and $\delta \vec{B}_\perp^a$ will not necessarily satisfy the 3D Maxwell equations by themselves. because $\vec{E}_\perp^a|_{\text{Linear Focus}}$ and $\vec{B}_\perp^a|_{\text{Linear Focus}}$ will not, in general, satisfy the 3D Maxwell equations by themselves either.

§5 Linear Equations of Motion without Space-Charge, Acceleration, and Momentum Spread

- Neglect:
- Space charge effects: $\phi \rightarrow 0$
 - Nonlinear Applied Focusing: \vec{B}^a, \vec{E}^a contain only linear focusing terms
 - Acceleration: $\beta_b \gamma_b = \text{const.}$
 - Momentum Spread Effects

Then the transverse particle equations of motion can be expressed in the form of Hill's equation:

$$x''(s) + K(s)x(s) = 0$$

$x = \perp$ coordinate, possibly formed from various coordinates (even longitudinal coordinates can be cast in this form at some levels of approx.)

$s =$ axial coordinate of ref. particle.

$x' = \frac{dx}{ds}$. primes denote derivatives with respect to s .

$K(s) =$ lattice focusing function.
(linear fields)

For a periodic lattice:

$$K(s + L_p) = K(s)$$

$L_p =$ Lattice Period.

For a ring, one also has the superperiodicity condition:

$$K(s + C) = K(s) \quad *$$

$C = N L_p$ $N =$ superperiodicity.

* This distinction matters when there are random error fields in the ring: (Repeat with super but not lattice period)

Transfer Matrix Form of Solution

The equation of motion is linear and the (unique) solution with

$$x(s=s_i) = x(s_i)$$

$$x'(s=s_i) = x'(s_i)$$

"initial" conditions

can be expressed in matrix form as:

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} \equiv \bar{M}(s|s_i) \begin{pmatrix} x(s_i) \\ x'(s_i) \end{pmatrix} \equiv \begin{pmatrix} C(s|s_i) & S(s|s_i) \\ C'(s|s_i) & S'(s|s_i) \end{pmatrix} \begin{pmatrix} x(s_i) \\ x'(s_i) \end{pmatrix}$$

where $C(s|s_i)$ and $S(s|s_i)$ are "cosine"-like and "sine"-like principal trajectories satisfying:

$$C''(s|s_i) + R(s)C(s|s_i) = 0; \quad C(s_i|s_i) = 1, \quad C'(s_i|s_i) = 0$$

$$S''(s|s_i) + R(s)S(s|s_i) = 0; \quad S(s_i|s_i) = 0, \quad S'(s_i|s_i) = 1$$

An important property of this linear motion is that the Wronskian is conserved:

$$W(s|s_i) \equiv \det \bar{M}(s|s_i) = \det \begin{vmatrix} C(s|s_i) & S(s|s_i) \\ C'(s|s_i) & S'(s|s_i) \end{vmatrix} \\ = C(s|s_i)S'(s|s_i) - C'(s|s_i)S(s|s_i) = 1$$

// Proof: (abbreviate notation)

$$\begin{aligned} -S(C'' + RC) &= 0 & \Rightarrow & CS'' - SC'' + R(CS - SC) = 0 \\ + C(S'' + RS) &= 0 & \Rightarrow & \frac{dW}{ds} = 0 \end{aligned}$$

$$\Rightarrow W(s) = W(s_i) = C_i S'_i - C'_i S_i = 1 \cdot 1 - 0 \cdot 0 = 1 \quad \checkmark //$$

// Example

For a continuously focused system:

$$R(s) = k_{p0}^2 = \text{const.}$$

$$x'' + k_{p0}^2 x = 0$$

$$C(s|s_i) = \cos[k_{p0}(s-s_i)]$$

$$S(s|s_i) = \frac{\sin[k_{p0}(s-s_i)]}{k_{p0}}$$

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \begin{pmatrix} \cos[k_{p0}(s-s_i)] & \frac{\sin[k_{p0}(s-s_i)]}{k_{p0}} \\ -k_{p0} \sin[k_{p0}(s-s_i)] & \cos[k_{p0}(s-s_i)] \end{pmatrix} \begin{pmatrix} x(s_i) \\ x'(s_i) \end{pmatrix}$$

$$\bar{W} = \cos^2[k_{p0}(s-s_i)] + \sin^2[k_{p0}(s-s_i)] = 1 \quad //$$

Stability

We now examine particle stability. First, \bar{M} must be the same in any period of a periodic lattice:

$$\bar{M}(s+L_p|s_i+L_p) = \bar{M}(s|s_i)$$

For a propagation distance $s-s_i$ satisfying

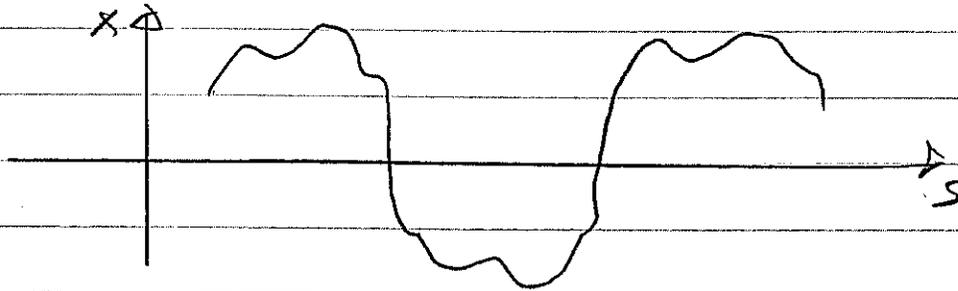
$$N L_p \leq s-s_i \leq (N+1) L_p$$

with N integer, the advance $\bar{M}(s|s_i)$ can be resolved as:

$$\bar{M}(s|s_i) = \overbrace{\bar{M}(s-NL_p|s_i)}^{\text{Residual}} \overbrace{\bar{M}(NL_p|s_i)}^{N \text{ period}} = \bar{M}(s-NL_p|s_i) \bar{M}(s_i+L_p|s_i)^N$$

For a particle to have a stable orbit $x(s)$ and $x'(s)$ should remain bounded on propagation through an

For a particle to have a stable orbit
 $x(s)$ and $x'(s)$ should remain bounded
 on propagation through an arbitrary number N of
 lattice periods



"energetic" \Rightarrow $H = \frac{1}{2}x'^2 + \frac{1}{2}kx^2 \sim$ large
 particle (but not constant)
 Quadratic:

Where: x small x' large
 x' small x large

This bounding is equivalent to requiring that the
 elements of \bar{M} remain bounded on propagation through
 N lattice periods.

This suggest the stability definition

$$\lim_{N \rightarrow \infty} \left| \left\{ \bar{M}(s_i + L_p | s_i) \right\}_{ij}^N \right| < \infty$$

where

$$\bar{A}_{ij} = {}_{ij} \text{ component of } \bar{M}$$

To analyze the stability condition - examine the eigenvectors/eigenvalues of \bar{M} for transport through one lattice period:

$$\bar{M}(s_i + L_p | s_i) \cdot \vec{E}(s_i) = \lambda \vec{E}(s_i)$$

$$\lambda = \text{eigenvalue}$$

$$\vec{E}(s_i) = \text{eigenvector}$$

Derive the 2 independent eigenvectors/eigenvalues through analysis of the characteristic equation:

Abbreviate:

$$\bar{M}(s_i + L_p | s_i) = \begin{pmatrix} C(s_i + L_p | s_i) & S(s_i + L_p | s_i) \\ C'(s_i + L_p | s_i) & S'(s_i + L_p | s_i) \end{pmatrix} \equiv \begin{pmatrix} C & S \\ C' & S' \end{pmatrix}$$

Nontrivial solutions exist when:

$$\det \begin{vmatrix} C - \lambda & S \\ C' & S' - \lambda \end{vmatrix} = \lambda^2 + (C + S')\lambda + (CS' - C'S) = 0$$

But:

$$\text{Wronskian Condition: } CS' - C'S = W = 1$$

$$\text{Trace Def: } C + S' = \text{Trace } \bar{M} \equiv 2 \cos \delta_0$$

Characteristic eqn:

$$\lambda^2 - 2\lambda \cos \delta_0 + 1 = 0$$

$$\lambda = \text{Trace } \bar{M}(s_i + L_p | s_i)$$

$$P = \sqrt{-1}$$

Solutions (2):

$$\lambda = \cos \delta_0 \pm \sqrt{\cos^2 \delta_0 - 1} = \cos \delta_0 \pm i \sin \delta_0$$

Denote 2 solutions as:

$$\lambda_{\pm} = \cos \delta_0 \pm i \sin \delta_0 = \text{eigenvalues}$$

$$\vec{E}_{\pm} = \text{corresponding eigenvectors}$$

Note:

$$\lambda_+ \lambda_- = 1$$

$$\lambda_- = \frac{1}{\lambda_+}$$

Denote an initial condition by

$$\vec{x}_i \equiv \begin{pmatrix} x(s_i) \\ x'(s_i) \end{pmatrix}$$

This initial condition can always be expanded in terms of the eigenvectors as:

$$\vec{x}_i = d_+ \vec{E}_+ + d_- \vec{E}_-$$

d_{\pm} constants

Then

$$\begin{aligned} \overline{M}^N \vec{x}_i &= d_+ \overline{M}^N \vec{E}_+ + d_- \overline{M}^N \vec{E}_- \\ &= d_+ \lambda_+^N \vec{E}_+ + d_- \lambda_-^N \vec{E}_- \end{aligned}$$

Thus, if λ_{\pm}^N remains bounded the motion is stable. This will always be the case if $|\lambda_+| \leq 1$, corresponding to δ_0 real with $|\cos \delta_0| \leq 1$, or equivalently

$$\frac{1}{2} \left| \text{Trace } \overline{M}(s_i + L_p | s_i) \right| = \frac{1}{2} \left| C(s_i + L_p | s_i) + S'(s_i + L_p | s_i) \right| = |\cos \delta_0| \leq 1$$

For stable motion.

// Example

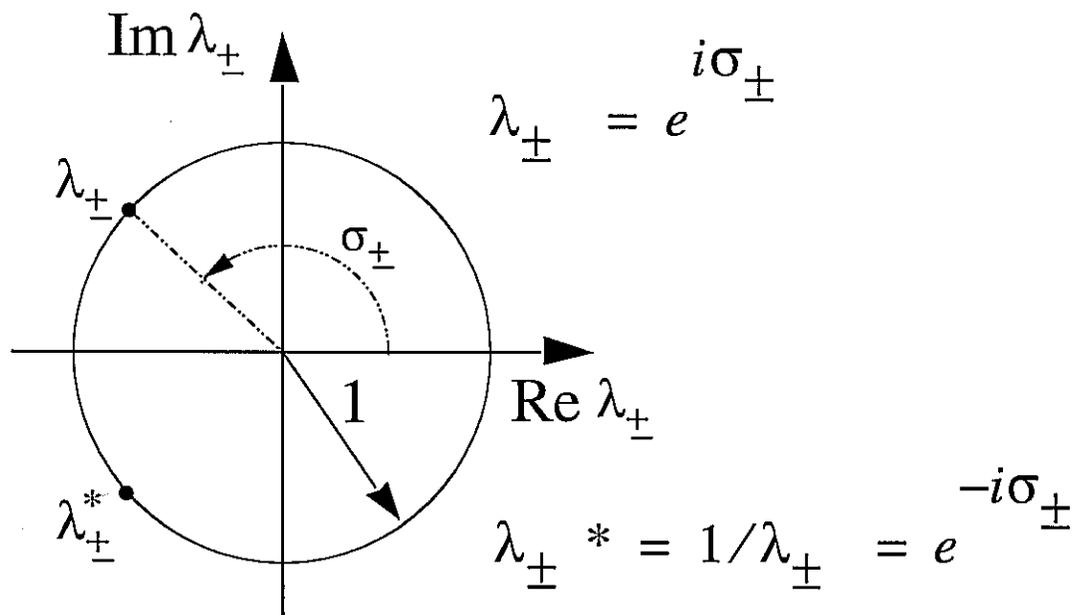
For continuous focusing:

$$\frac{1}{2} \left| \text{Trace } [M(s_i + L_p | s_i)] \right| = |\cos [k_{po} L_p]| \leq 1$$

Motion is stable for all k_{po} real.

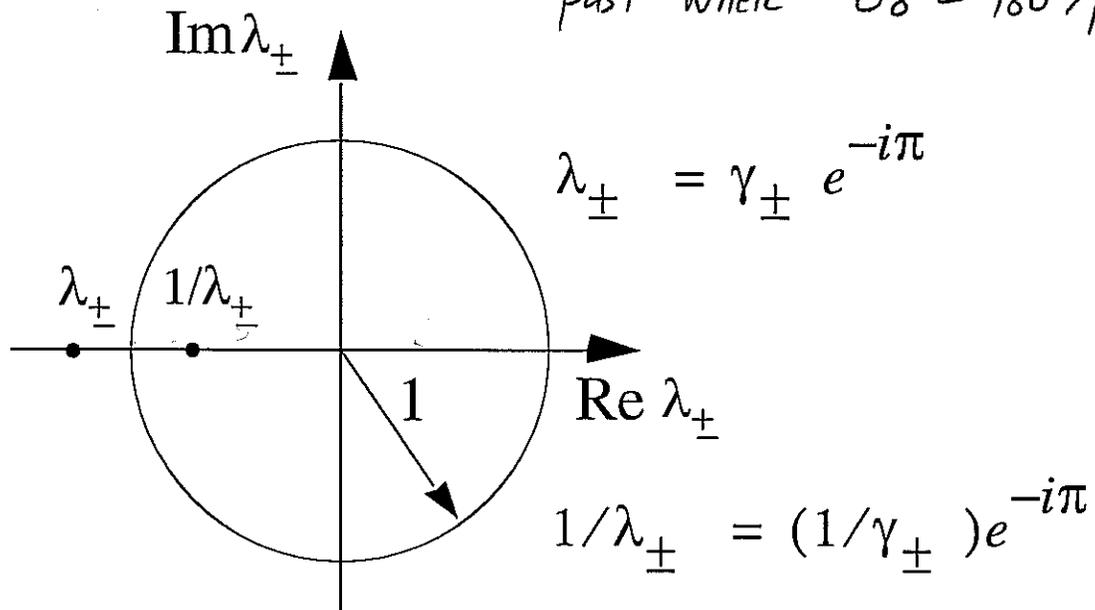
Possible Eigenvalue Symmetries for 2x2 Transfer Matrices

a) Stable Occurs for $0 \leq \sigma_0 \leq 180^\circ / \text{period}$



b) Unstable, Lattice Resonance

Occurs in bands after focusing strengths are increased past where $\sigma_0 = 180^\circ / \text{period}$.



For more info see A. Dragt, "Lectures on Nonlinear Orbit Dynamics"

in "Physics of High Energy Accelerators" AIP Conf. Proc. No. 87 (AIP, New York, 1982) p. 147

In a periodic focusing system this condition will place restrictions on the lattice structure that are generally thought of in terms of phase advance limits. Accelerator lattices are almost always tuned for single particle stability. Because even for intense beam machines, when image charges are neglected, the beam centroid obeys single particle equations of motion and instability would result in undesirable, large beam centroid excursions.

- Space charge effects can introduce additional stability constraints on the lattice
 - Envelope modes may be unstable (lowest order collective mode)
 - Higher order space charge modes may be unstable.
- Nonlinear focusing terms in the applied field and machine construction errors etc. may also induce instabilities.

In the Homework problems, a simple example of a stability limit for a thin lens "FODO" periodic focusing channel is analyzed.

Later, we will show that for stable orbits that \bar{J}_0 can be interpreted as the phase-advance of single particle oscillations moving in the linear applied field.

§6 Floquet's Theorem and the Phase-Amplitude Form of the Particle Orbit

Hill's equation:

$$x''(s) + R(s)x(s) = 0$$

For periodic $R(s)$:

$$R(s+L_p) = R(s)$$

Floquet's Theorem:

$x(s)$ has two linearly independent solutions that can be expressed as:

$$x_1(s) = W(s) e^{i\mu s}$$
$$x_2(s) = W(s) e^{-i\mu s}$$
$$\mu = \frac{1}{2} \text{Tr} \cdot \bar{M}(s_i+L_p|s_i)$$
$$= \cos \delta_0$$
$$= \text{const}$$

where $W(s)$ is a periodic function:

$$W(s+L_p) = W(s)$$

characteristic exponent

- Strictly speaking, theorem as written only applies when $\bar{M}(s_i+L_p|s_i)$ has non-degenerate eigenvalues. But a similar theorem applies to the degenerate case.
- Similar theorem is also valid for non-periodic $R(s)$.

As a consequence of Floquet's theorem, any stable or nondegenerate unstable solution can be expressed in phase-amplitude form:

Phase-
Amplitude
Form

$$x(s) = A(s) \cdot \cos \Psi(s)$$

$$A(s) = \text{Amplitude}$$

$$A(s+Lp) = A(s)$$

$$\Psi(s) = \text{Phase}$$

Derive eqns of motion in terms of A, Ψ :

$$x = A \cos \Psi$$

$$x' = A' \cos \Psi - A \Psi' \sin \Psi$$

$$x'' = A'' \cos \Psi - 2A' \Psi' \sin \Psi - A \Psi'' \sin \Psi - A \Psi'^2 \cos \Psi$$

substitute in Hill's equation:

$$x'' + R \cdot x = [A'' + RA - A \Psi'^2] \cos \Psi - [2A' \Psi' + A \Psi''] \sin \Psi = 0$$

We are free to introduce a constraint between A and Ψ .

- 2 functions A, Ψ ; residual freedom allows an additional constraint

Take

$$2A' \Psi' + A \Psi'' = 0$$

$$\Rightarrow \text{coeff of } \sin \Psi = 0$$

Then, since $\cos \Psi \neq 0$ for all Ψ , the transformed equation of motion then gives:

$$A'' + RA - A \Psi'^2 = 0$$

$$\Rightarrow \text{coeff of } \cos \Psi = 0$$

The equation $ZA'\psi' + A\psi'' = 0$ (coeff. of $\sin\psi$) can be integrated

$$ZA'\psi' + A\psi'' = (A^2\psi')'/A = 0 \quad A \neq 0$$

$$\Rightarrow (A^2\psi')' = 0$$

$$\Rightarrow \boxed{A^2\psi' = \text{const}}$$

One commonly rescales $A(s)$ such that this constant can be taken to be unity. Take

$$\boxed{A(s) = A_0 W(s) \quad ; \quad A_0 = \text{const}}$$

$$\boxed{W^2\psi' \equiv 1}$$

Then $W^2\psi' = 1$ can be equivalently expressed as

$$\boxed{\psi(s) = \psi_0 + \int_{s_0}^s \frac{d\tilde{s}}{W^2(\tilde{s})}}$$

$$\psi_0 = \text{const.}$$

= Initial phase

Using $W^2\psi' = 1$ and $A = A_0 W$ in the other equation $A'' + RA - A\psi'' = 0$ (coeff of $\cos\psi$) then gives:

$$\boxed{W'' + RW - \frac{1}{W^3} = 0}$$

where W is the periodic solution with period L_p , i.e.,

$$\boxed{W(s + L_p) = W(s)}$$

Summary: Phase Amplitude Form

$$x(s) = A_i w(s) \cos \psi(s)$$

where:

$$\begin{cases} w''(s) + R(s)w(s) - \frac{1}{w^3(s)} = 0 \\ w(s+L_p) = w(s) \end{cases}$$

$$\psi'(s) = \frac{1}{w^2(s)} \quad ; \quad \psi(s) = \psi_i + \int_{s_i}^s \frac{d\tilde{s}}{w^2(\tilde{s})}$$

$$\equiv \psi_i + \Delta\psi(s)$$

In this phase-amplitude formulation, the particle trajectory in $x-x'$ phase-space is:

$$\begin{aligned} x(s) &= A_i w(s) \cos \psi(s) \\ x'(s) &= A_i w'(s) \cos \psi(s) - \frac{A_i}{w(s)} \sin \psi(s) \end{aligned}$$

Here, we have used $\psi' = 1/w^2$. The initial particle coordinates $x(s_i)$ and $x'(s_i)$ at $s=s_i$ are related to the amplitude and phase parameters at $s=s_i$, A_i and ψ_i , by:

$$\begin{aligned} x(s_i) &= A_i w_i \cos \psi_i \\ x'(s_i) &= A_i w_i' \cos \psi_i - \frac{A_i}{w_i} \sin \psi_i \end{aligned}$$

$$w_i \equiv w(s_i) \quad ; \quad w_i' \equiv w'(s_i)$$

-or-

$$A_i \cos \psi_i = x(s_i) / w_i$$

$$A_i \sin \psi_i = x(s_i) w_i' - x'(s_i) w_i$$

Some points

- 1) The sign of $W(s)$ can be taken to be positive definite

$$W(s) > 0$$

"Proof":

Let $W(s)$ be positive at some value of s . Then

$$W'' + \rho W - \frac{1}{W^3} = 0$$

insures that W can never vanish or change sign since as W becomes small, $W'' \approx 1/W^3 > 0$ can become arbitrarily large, thereby turning W back to the positive direction before W can vanish or reach negative values

- Sign choice removes ambiguity in relating initial conditions $X(s_i)$ and $X'(s_i)$ to A_i and Ψ_i

- 2) $W(s)$ is a unique periodic function.

- Can be proved using later connections between W and principal orbits.
- W can be regarded as a special, periodic function characterizing the lattice.

- 3) The amplitude parameters:

$$W_i = W(s_i)$$

$$W_i' = W'(s_i)$$

depend only on the periodic lattice properties and are independent of the particle initial conditions.

4) The phase advance

$$\Delta\psi(s) \equiv \int_{s_i}^s \frac{d\tilde{s}}{W^2(\tilde{s})}$$

will, in contrast, depend on s_i . However the phase advance through one Lattice period

$$\Delta\psi(s_i + L_p) = \int_{s_i}^{s_i + L_p} \frac{d\tilde{s}}{W^2(\tilde{s})}$$

will be independent of s_i since W is a periodic function with period L_p .

- Will show that

$$\Delta\psi(s_i + L_p) = \cos \delta_0$$

is the undepressed particle phase advance.

5) $W(s)$ has dimension [meters^{1/2}]

- Inconvenience of this motivate use of an alternative "betatron" function:

$$\beta(s) \equiv W^2(s)$$

with dimension [meters], (see §8)

6) On the surface, what we have done: Transform a linear Hill's equation to nonlinear equations for w and ψ via the phase amplitude method seems insane. Why?

- Method will help identify useful invariant (see §7)
- Decoupling of initial conditions enabled by phase-amplitude method will help simplify understanding of bundles of particles.

The principal functions in the transfer matrix analysis of the particle orbit can be expressed in phase amplitude form as:

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \bar{M}(s|s_i) \begin{pmatrix} x(s_i) \\ x'(s_i) \end{pmatrix} = \begin{pmatrix} C(s|s_i) & S(s|s_i) \\ C'(s|s_i) & S'(s|s_i) \end{pmatrix} \begin{pmatrix} x(s_i) \\ x'(s_i) \end{pmatrix}$$

Some algebra identities (in the problem sets):

$$C(s|s_i) = \frac{W(s)}{W_i} \cos \Delta\psi(s) - W_i' W(s) \sin \Delta\psi(s)$$

$$S(s|s_i) = W_i W(s) \sin \Delta\psi(s)$$

$$C'(s|s_i) = \left(\frac{W'(s)}{W_i} - \frac{W_i'}{W(s)} \right) \cos \Delta\psi(s) - \left(\frac{1}{W_i W(s)} + W_i' W'(s) \right) \sin \Delta\psi(s)$$

$$S'(s|s_i) = \frac{W_i}{W(s)} \cos \Delta\psi(s) + W_i W'(s) \sin \Delta\psi(s)$$

$$\text{where } \Delta\psi(s) \equiv \int_{s_i}^s \frac{ds}{W^2(s)} \quad \begin{array}{l} W_i = W(s=s_i) \\ W_i' = W'(s=s_i) \end{array}$$

Some manipulation (Appendix A) also shows that $W(s)$ is related to the principal orbit functions as:

$$W^2(s) = \beta(s) = \frac{\sin \delta_0 \cdot S'(s|s_i)}{S'(s_i+L_p|s_i)} + \frac{S'(s_i+L_p|s_i)}{\sin \delta_0} \left[\frac{C(s|s_i) + (\cos \delta_0 - C(s_i+L_p|s_i)) S'(s|s_i)}{S'(s_i+L_p|s_i)} \right]^2$$

$$\delta_0 \equiv \int_{s_i}^{s_i+L_p} \frac{ds}{W^2(s)} = \text{phase advance}$$

This formula for $W^z(s)$ in terms of principal orbits is very useful

- Δ_0 (phase advance, see following pages) is generally specified for the lattice.
- Shows that W can be constructed with two principal orbit integrations over one lattice period
 - Integrations can be done numerically for $G(s|s_i)$ and $S(s|s_i)$.
 - No root finding for appropriate "initial" conditions is needed to construct $W(s)$ periodic
 - s_i can be anywhere in the lattice period and $W(s)$ will be independent of the specific choice of s_i
- The form of W^z suggests an underlying Courant-Snyder invariant (see Appendix A and §7).
- $W^z = \beta$ can be used to calculate the maximum beam particle excursions in the absence of space-charge. (see §8)
 - Very useful in machine design
 - Exploits Courant-Snyder Invariant (see §7)

Undepressed Particle Phase Advance

Now we can now connect δ_0 for a stable orbit to the advance in phase Ψ of the orbit through one lattice period:

$$\begin{aligned}\cos \delta_0 &\equiv \frac{1}{2} \text{Trace } \bar{M}(s_i + L_p | s_i) \\ &= \frac{1}{2} \left[C(s_i + L_p | s_i) + \beta'(s_i + L_p | s_i) \right] \\ &= \frac{1}{2} \left(\frac{W(s_i + L_p)}{W_i} + \frac{W_i}{W(s_i + L_p)} \right) \cos \Delta\Psi(s_i + L_p) \\ &\quad + \frac{1}{2} \left(W_i W'(s_i + L_p) - W_i' W(s_i + L_p) \right) \sin \Delta\Psi(s_i + L_p)\end{aligned}$$

By periodicity:

$$W(s_i + L_p) = W_i$$

$$W'(s_i + L_p) = W_i'$$

\Rightarrow

$$\text{coefficient of } \cos \Delta\Psi(s_i + L_p) = 1$$

$$\text{coefficient of } \sin \Delta\Psi(s_i + L_p) = 0$$

$$\cos \delta_0 = \cos \Delta\Psi(s_i + L_p) = \frac{1}{2} \text{Trace } \bar{M}(s_i + L_p | s_i)$$

Thus δ_0 is identified as the phase-advance of a stable particle orbit through one lattice period.

$$\delta_0 \equiv \Delta\Psi(s_i + L_p) = \int_{s_i}^{s_i + L_p} \frac{ds}{W^2(s)}$$

Undepressed Particle
Phase Advance.

Note that δ_0 is independent of s_i since $W(s)$ is periodic.

The stability criteria $\frac{1}{2} |\text{Trace } \bar{M}(s_i + L_p | s_i)| = |\cos \delta_0| \leq 1$ is now connected to the phase advance of the particle through one period of the focusing lattice.

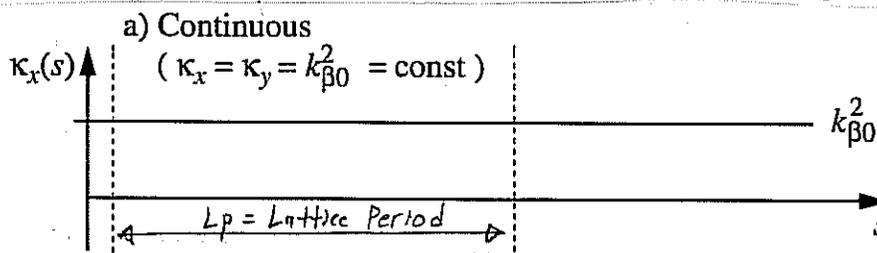
The phase advance is an extremely useful dimensionless measure to characterize the focusing strength of lattices. Much of conventional accelerator physics centers on focusing strength and suppression of resonance effects. The phase advance is a natural parameter to employ in these situations to present results in a readily interpretable and generalizable manner.

We now present phase advance formulas for $\bar{\sigma}_0$ for 3 classes of ^{periodic} lattices with piecewise constant $\kappa_x(s)$ and $\kappa_y(s)$

- Continuous Focusing
- Solenoidal Focusing
- Quadrupole Doublet focusing

These formulas are derived from transfer matrix analysis and direct integration of the equations of motion:

1/ Continuous Focusing

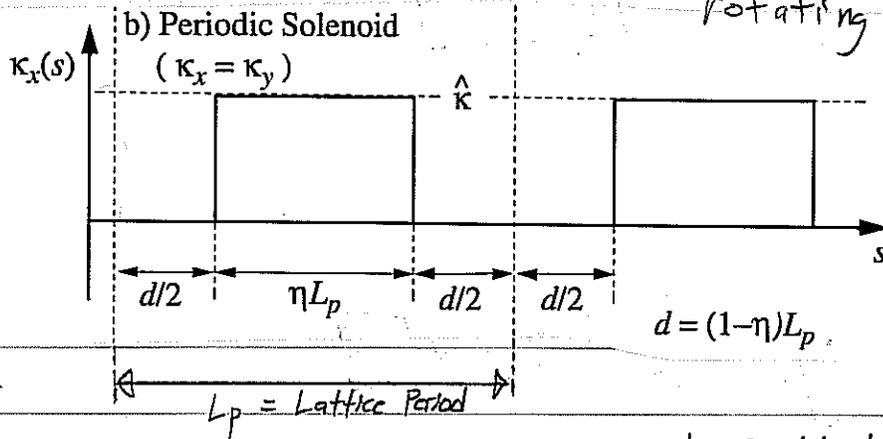


$$\bar{\sigma}_0 = k_{\beta 0} L_p$$

The "lattice period" here is an arb. length for phase accumulation

b) Periodic Solenoidal Focusing

Phase advance must be interpreted in the rotating Larmor frame. (see earlier lectures.)



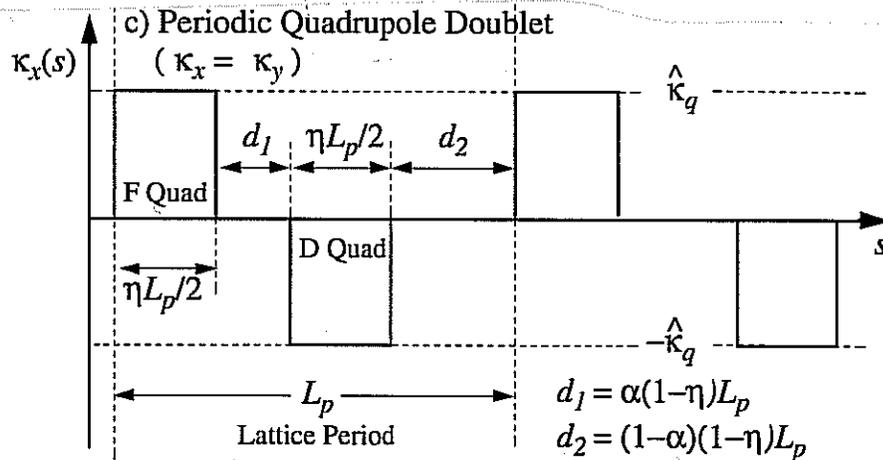
$d = \text{Drift length}$

$\eta \in [0, 1] = \text{occupancy of solenoid in period.}$

$$\cos \sigma_0 = \cos(2\Theta) - \frac{1-\eta}{\eta} \Theta \sin(2\Theta)$$

$$\Theta \equiv \sqrt{\hat{k}} \eta L_p / 2$$

c) Periodic Quadrupole Doublet Lattice



$$\begin{aligned} \cos \sigma_0 = & \cos \Theta \cosh \Theta \\ & + \frac{1-\eta}{\eta} \Theta (\cos \Theta \sinh \Theta - \sin \Theta \cosh \Theta) \\ & - 2\alpha(1-\alpha) \frac{(1-\eta)^2}{\eta^2} \Theta^2 \sin \Theta \sinh \Theta \end{aligned}$$

$$\Theta \equiv \sqrt{|\hat{k}_q|} \eta L_p / 2$$

d_1, d_2 drift lengths
 $\eta \in (0, 1] = \text{occupancy of quads in period}$
 $\alpha \in [0, 1] = \text{synchrotron parameter}$

α measures how close focusing (F) and defocusing (D) quadrupoles are to each other in the lattice.

$$\alpha = 0: \quad d_1 = 0, \quad d_2 = (1-\eta)L_p$$

$$\alpha = 1: \quad d_1 = (1-\eta)L_p, \quad d_2 = 0$$

$$\alpha = \frac{1}{2} \Rightarrow d_1 = d_2 = (1-\eta)\frac{L_p}{2} \equiv \text{"FODO" lattice,}$$

with equal drifts.

The range $\alpha \in [\frac{1}{2}, 1]$ can be mapped to $\alpha \in [0, \frac{1}{2}]$ by relabeling lattice quantities. Therefore, take:

$$\alpha \in [0, \frac{1}{2}]$$

Thin Lens limit

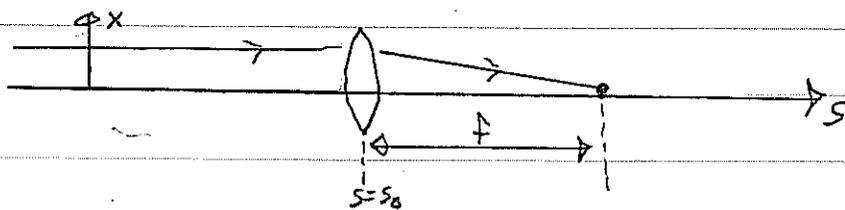
Convenient to simply understand scaling

$$\mu_x(s) = \frac{1}{f} \delta(s - s_0)$$

$s_0 =$ optic location.

$f =$ const, focal length.

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s=s_0^+} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{s=s_0^-}$$



The thin lens limit of "thick" solenoids and quadrupoles can be obtained by taking

Solenoids: $\hat{K} \equiv \frac{1}{\eta^2 L_p}$; then $\lim_{\eta \rightarrow 0}$

Quadrupoles: $\hat{K}_q \equiv \frac{2}{\eta^2 L_p}$; then $\lim_{\eta \rightarrow 0}$

This obtains:

$$\cos \sigma_0 = \begin{cases} 1 - \frac{1}{2} \frac{L_p}{f}, & \text{thin-lens solenoids,} \\ 1 - \frac{\alpha}{2} (1 - \alpha) \left(\frac{L_p}{f}\right)^2, & \text{thin-lens quadrupoles.} \end{cases}$$

This formula can also be derived directly from thin lens transfer matrices:

$$\cos \sigma_0 = \frac{1}{2} \text{Tr} \begin{bmatrix} 1 & L_p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} = 1 - \frac{1}{2} \frac{L_p}{f},$$

Solenoid lattice

$$\begin{aligned} \cos \sigma_0 &= \frac{1}{2} \text{Tr} \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \begin{bmatrix} 1 & \alpha L_p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{bmatrix} \\ &\times \begin{bmatrix} 1 & (1 - \alpha) L_p \\ 0 & 1 \end{bmatrix} = 1 - \frac{\alpha}{2} (1 - \alpha) \left(\frac{L_p}{f}\right)^2, \end{aligned}$$

Quadrupole doublet lattice.

In many cases it is also desirable to have simple formulas for design that relate magnet parameters to σ_0 . For example, if the formula for $\cos \sigma_0$ is expanded to leading order in $\alpha = \sqrt{|\hat{K}_q|} \cdot \eta L_p / 2$ for quadrupole doublet focusing, we obtain:

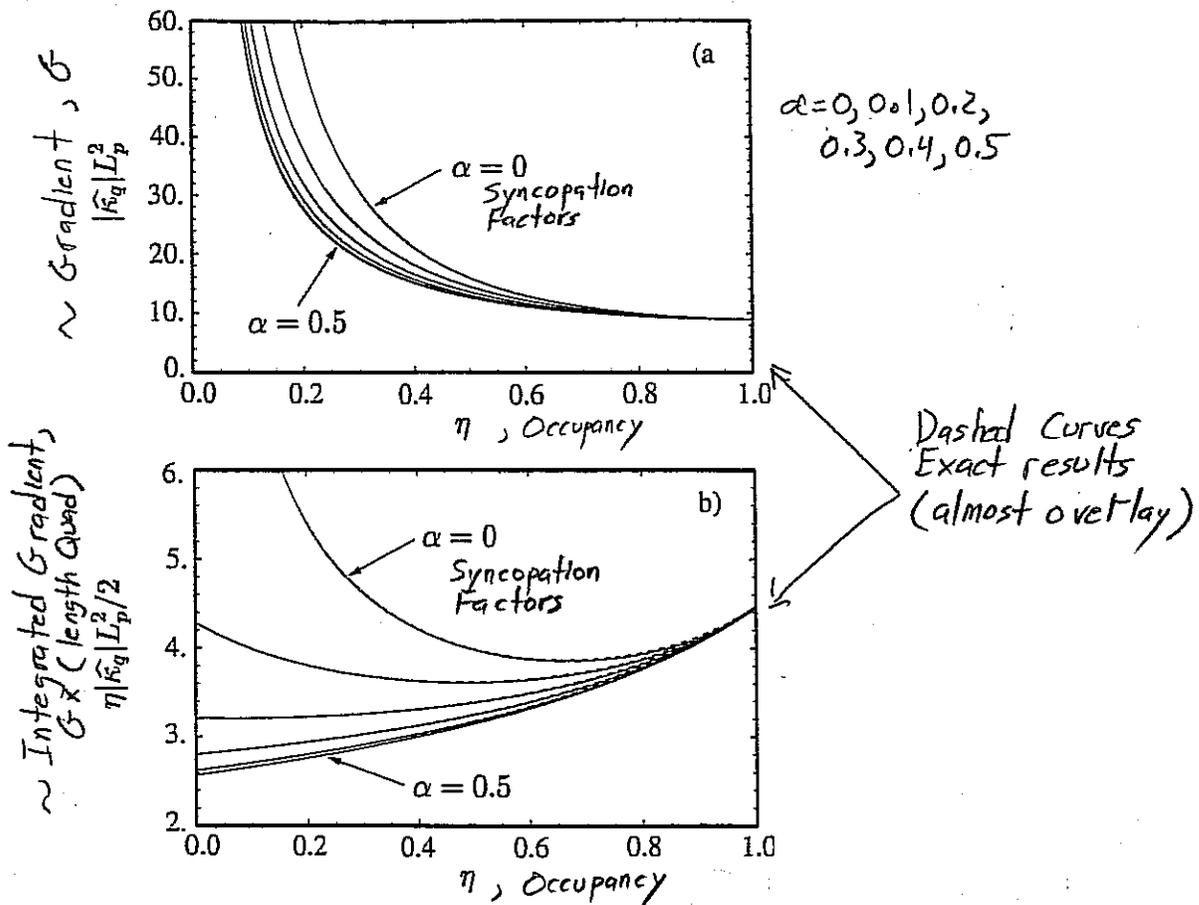
$$\cos \sigma_0 = 1 - \frac{(\eta \hat{K}_q L_p^2)^2}{32} \left[\left(1 - \frac{2}{3}\eta\right) - 4 \left(\alpha - \frac{1}{2}\right)^2 (1 - \eta)^2 \right].$$

where

$$\hat{K}_q = \begin{cases} \frac{\hat{G}}{[B\rho]} & ; \text{Magnetic Quadrupoles} \\ \frac{\hat{G}}{\beta_0 c [B\rho]} & ; \text{Electric Quadrupoles} \end{cases} \quad \hat{G} = \text{Field Gradient.}$$

Plot eqn:

For $\sigma_0 = 80^\circ / \text{period}$ Quadrupole



96 Appendix A: Calculation of $W(s)$ from Principal Orbit Functions

Evaluate transfer matrix elements through one lattice period using:

$$W(s_i + L_p) = W_i$$

$$W'(s_i + L_p) = W_i'$$

and

$$\Delta\psi(s_i + L_p) = \int_{s_i}^{s_i + L_p} \frac{ds}{W^2(s)} = \delta_0$$

to obtain:

$$C(s_i + L_p | s_i) = \cos \delta_0 - W_i W_i' \sin \delta_0$$

$$S(s_i + L_p | s_i) = W_i^2 \sin \delta_0$$

$$C'(s_i + L_p | s_i) = -\left(\frac{1}{W_i^2} + W_i W_i'\right) \sin \delta_0$$

$$S'(s_i + L_p | s_i) = \cos \delta_0 + W_i W_i' \sin \delta_0$$

giving:

$$W_i = \sqrt{\frac{S'(s_i + L_p | s_i)}{\sin \delta_0}}$$

$$W_i' = \frac{\cos \delta_0 - C(s_i + L_p | s_i)}{\sqrt{S'(s_i + L_p | s_i) \sin \delta_0}}$$

or using $\beta = W^2$ and $\beta' = 2WW'$

$$\beta_i = W_i^2 = \frac{S'(s_i + L_p | s_i)}{\sin \delta_0}$$

$$\beta_i' = 2W_i W_i' = \frac{2(\cos \delta_0 - C(s_i + L_p | s_i))}{\sin \delta_0}$$

Next, calculate W from the elements

$$\frac{S'}{W_i W} = \sin \Delta \Psi$$

Here

$$S' \equiv S'(s|s_i)$$

etc.

$$\frac{W_i C}{W} + \frac{W_i' S'}{W} = \cos \Delta \Psi$$

square:

$$\frac{S'^2}{W_i^2 W^2} + \left(\frac{W_i C}{W} + \frac{W_i' S'}{W} \right)^2 = 1$$

- See p 7: This structure, not accidentally, has a Courant-Snyder invariant form.

$$\Rightarrow W^2 = \frac{S'^2}{W_i^2} + \left(W_i C + W_i' S' \right)^2$$

Use W_i and W_i' identified and write out:

$$W^2(s) = \beta^2(s) = \frac{\sin^2 \Delta_0}{S'(s_i + L_p | s_i)} \frac{S'^2(s | s_i)}{S'(s_i + L_p | s_i)} + \frac{S'(s_i + L_p | s_i)}{\sin \Delta_0} \left[C(s | s_i) + \frac{(\cos \Delta_0 - C(s_i + L_p | s_i)) S'(s | s_i)}{S'(s_i + L_p | s_i)} \right]^2$$

- Formula shows that for given Δ_0 (used to specify lattice focus strength) that $W(s)$ is given by two linear principal orbits calculated over one lattice period.
- Easy to do numerically!

An alternative way to calculate $W(s)$ or $\beta(s)$ is as follows:

$$C(s+L_p|s) = \cos \delta_0 - W(s)W'(s) \sin \delta_0$$

$$S'(s+L_p|s) = W^2(s) \sin \delta_0$$

Thus

$$W^2(s) = \beta(s) = \frac{S'(s+L_p|s)}{\sin \delta_0}$$

$S(s+L_p|s)$

is M_{12} element
of $\bar{M}(s+L_p|s)$

Direct application of this formula requires calculation of $S(s+L_p|s)$ at every value of s desired over the period. The previous method requires one calculation of $C(s|s_i)$ and $S(s|s_i)$ for $s_i \leq s \leq s_i+L_p$ for any value of s_i . Thus, although the previous formula is simpler to apply in practice, yet another alternative, is to calculate $W^2(s)$ using Matrix algebra and symmetry:

Advance interval:



$$M_{11}(s+L_p|s) = \sin \delta_0 W^2(s) = \sin \delta_0 \beta(s)$$

Alternative
expression
of above.

But

$$\begin{aligned} \bar{M}(s+L_p|s) &= \bar{M}(s+L_p|s_i+L_p) \bar{M}(s_i+L_p|s) \bar{M}(s|s_i) \quad // \text{I} \\ &= \bar{M}(s|s_i) \bar{M}(s_i+L_p|s) \underbrace{[\bar{M}(s|s_i) \bar{M}^{-1}(s|s_i)]}_{= \bar{M}(s_i+L_p|s_i)} \end{aligned}$$

-or-

$$\bar{M}(s+L_p|s) = \bar{M}(s|s_i) \bar{M}(s_i+L_p|s_i) \bar{M}^{-1}(s|s_i)$$

Since this uses only elements $\bar{M}(s|s_i)$ for $s_i \leq s \leq s_i+L_p$ on the RHS, it shows that $\bar{M}(s+L_p|s)$ can be calculated from the evaluation of $\bar{M}(s|s_i)$ over one period with s_i anywhere in the period.

Only the M_{12} element will be needed to calculate $W'(s)$.

If we take

$$\bar{M} = \begin{pmatrix} C & S' \\ C' & S \end{pmatrix}$$

$$\det \bar{M} = 1$$

(Wronskian condition)

$$\Rightarrow \bar{M}^{-1} = \begin{pmatrix} S' & -C' \\ -S & C \end{pmatrix}$$

Using in the eqn above:

$$\bar{M}(s+L_p|s) = \begin{pmatrix} C(s|s_i) & -S'(s|s_i) \\ C'(s|s_i) & -S(s|s_i) \end{pmatrix} \begin{pmatrix} C(s_i+L_p|s_i) & S(s_i+L_p|s_i) \\ C'(s_i+L_p|s_i) & S'(s_i+L_p|s_i) \end{pmatrix} \\ \cdot \begin{pmatrix} S'(s|s_i) & -C'(s|s_i) \\ -S(s|s_i) & C(s|s_i) \end{pmatrix}$$

The \bar{M}_{12} element of this can be explicitly calculated.

Using the phase advance definition and the Wronskian, some algebra (see Lund, Chilton, and Lee, "Phys. Rev. Special Topics - Accel. and Beams, 2006") obtains the same formula previously derived.

§ 7 The Courant-Snyder Invariant and Single-Particle Emittance

Constants of the motion simplify interpretation of dynamics in physics.

// Example Review: Continuous Focusing System \leftrightarrow simple harmonic oscillator.

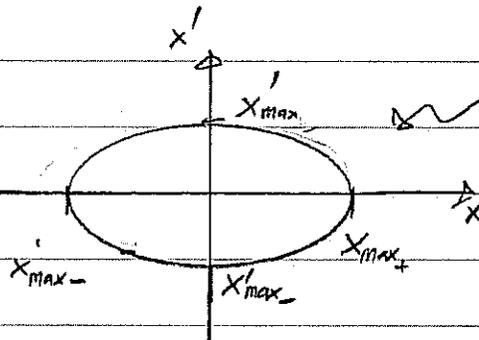
$$x'' + k_{FO}^2 x = 0 \quad ; \quad k_{FO}^2 = \text{const.}$$

$$\Rightarrow \boxed{H = \frac{x'^2}{2} + \frac{k_{FO}^2}{2} x^2 = \text{const.}} \quad \text{Conserved Hamiltonian}$$

Particle motion is constrained to move on an ellipse in $x-x'$ phase space with max extents

$$x_{\text{max}} \Rightarrow x' = 0 \quad ; \quad x_{\text{max}\pm} = \pm \sqrt{2H} / k_{FO}$$

$$x'_{\text{max}} \Rightarrow x = 0 \quad ; \quad x'_{\text{max}\pm} = \pm \sqrt{2H}$$



Location of particle on ellipse set by initial conditions. But all initial conditions yield the same shape ellipse thereby simplifying understanding of phase-space evolution.

Question

For Hill's equation:

$$x''(s) + K(s)x(s) = 0$$

Is there a quadratic invariant that can aid interpretation of the dynamics? Yes! Courant-Snyder Invariant!

Courant-Snyder Invariant

The Courant-Snyder Invariant of Hill's equation is extremely important in accelerator physics.

- Helps interpretation of dynamics
- Named after its discoverers, Courant and Snyder, who in a seminal (and very elegant!) paper simultaneously developed strong (AG) focusing and constructed/exploited this invariant to understand AG focusing.

Courant and Snyder, "Theory of the Alternating Gradient Synchrotron". *Annals of Physics* 3; 1 (1958).

— Christofilos also understood AG focusing via a more heuristic analysis.

The phase-amplitude method described in §6 makes the construction easy!

$$x(s) = A_i \cdot w(s) \cos \psi(s)$$

$$x'(s) = A_i w'(s) \cos \psi(s) + \frac{A_i}{w(s)} \sin \psi(s)$$

where

$$w'' + K w - \frac{1}{w^3} = 0$$

Re-arrange the phase-amplitude trajectory equations:

$$\frac{x}{w} = A_i \cos \psi$$

$$wx' - w'x = A_i \sin \psi$$

Add the squares to obtain the Courant-Snyder invariant

Courant-Snyder
Invariant

$$\left(\frac{x}{w}\right)^2 + (wx' - w'x)^2 = A_i^2 (\cos^2 \psi + \sin^2 \psi) \\ = A_i^2 = \text{const}$$

- Quadratic structure in x, x' defines a rotated ellipse in $x-x'$ phase-space.

- $w^2 \left(\frac{x}{w}\right)' = wx' - w'x$, so we can alternatively express the Courant-Snyder invariant as:

$$\left(\frac{x}{w}\right)^2 + \left[w^2 \left(\frac{x}{w}\right)'\right]^2 = \text{const.}$$

To better interpret the Courant-Snyder invariant, expand while isolating terms in x^2 , xx' , and x'^2

$$\left(\frac{1}{w^2} + w'^2\right)x^2 + 2(-ww')xx' + w^2x'^2 = A_i^2 = \text{const.}$$

The coefficients of x^2 , xx' , and x'^2 are functions of the lattice only and do not involve particle initial conditions. It is conventional to label these coefficients using "Twiss parameter" notation as:

$$\beta(s) \equiv W^2(s)$$

$$\alpha(s) \equiv -W(s)W'(s)$$

$$\gamma(s) \equiv \frac{1}{W^2(s)} + W'(s)^2 = \frac{1 + \alpha^2(s)}{\beta(s)}$$

Then the Courant-Snyder Invariant is expressed as:

$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = A_0^2 = \text{const.}$$

The "area" of this ellipse in xx' phase space is:

$$\text{Area} = \int_{\text{ellipse}} dx dx' = \frac{\pi A_0^2}{\sqrt{\gamma\beta - \alpha^2}} = \pi A_0^2 \equiv \pi \epsilon$$

Analytic Geometry Formula.

where ϵ is the single particle emittance which corresponds to the phase space area enclosed by the particle orbit divided by π .

// Aside Units

The definition of emittance is not unique. Some prefer to call $\pi\epsilon$ the emittance. Some write emittance values in units of mm-mrad with a " π " in the units as a reminder, i.e., $\epsilon = 10 \pi$ -mm-mrad, etc. Use caution to understand the convention used! //

From these identifications:

$$\gamma X^2 + 2\alpha XX' + \beta X'^2 = \epsilon = \text{const.}$$

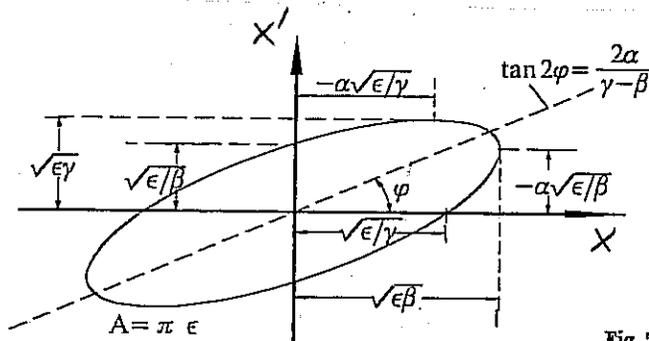


Fig. 5.22. Phase space ellipse

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From Wiedemann

Properties of Courant-Snyder Invariant

- The ellipse will rotate and change shape as the particle advances through the focusing lattice. But the area of the ellipse will not change. ($\pi\epsilon = \text{const}$).
- The location of the particle on the ellipse and the size of the ellipse depends on the initial conditions of the particle.
- The orientation of the ellipse is independent of the particle initial conditions. All particles move on nested ellipses.
- The Courant-Snyder invariant is quadratic in the phase-space coordinates, but it is not the transverse particle energy (which is not conserved).

// Example Particle Hamiltonian and energy of a kicked oscillator

$$x'' + K(s)x = 0 \quad ; \quad K(s) \neq \text{const.}$$

$$H = \frac{x'^2}{2} + \frac{K X^2}{2} \neq \text{const}$$

to see this:

$$\frac{dx'}{ds} = \frac{\partial H}{\partial x'} = x'$$

$$\frac{dx}{ds} = -\frac{\partial H}{\partial x} = -KX$$

$$\frac{dH}{ds} = \frac{\partial H}{\partial s} + \frac{\partial H}{\partial x} \frac{dx}{ds} + \frac{\partial H}{\partial x'} \frac{dx'}{ds} = \frac{\partial H}{\partial s} = \frac{K' X^2}{2} \neq \text{const.}$$

The energy of a kicked oscillator is not conserved. Do not confuse the energy of an oscillator with the Courant-Snyder invariant. Only in a continuous focusing system (simple harmonic oscillator) will they be simply related:

$$K(x) = k_{\beta 0}^2 = \text{const} \quad \text{continuous focusing}$$

$$\Rightarrow H = \frac{x'^2}{2} + \frac{k_{\beta 0}^2 X^2}{2} = \text{const.}$$

$$W'' + k_{\beta 0}^2 W - \frac{1}{W^3} = 0$$

$$\Rightarrow W = \sqrt{\frac{1}{k_{\beta 0}}} = \text{const} \quad \text{solution.}$$

$$\beta(s) = W^2 = 1/k_{\beta 0} = \text{const}$$

$$d(s) = -WW' = 0$$

$$\gamma(s) = \frac{1}{W^2} + W'^2 = k_{\beta 0} = \text{const.}$$

$$\varepsilon = k_{\beta 0} X^2 + \frac{X'^2}{k_{\beta 0}} = \frac{2}{k_{\beta 0}} \left(\frac{1}{2} X'^2 + \frac{k_{\beta 0}^2 X^2}{2} \right) = \text{const.}$$

$$= 2H/k_{\beta 0} = \text{const.}$$

Aside: Lattice Maps

The Courant-Snyder invariant helps us understand the phase space evolution of the particles. Knowing how the ellipse transforms (twists and rotates without changing area) is equivalent to knowing the dynamics of a bundle of particles

General S :

$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = \epsilon$$

Initial $S = S_i$:

$$\gamma_i x_i^2 + 2\alpha_i x_i x_i' + \beta_i x_i'^2 = \epsilon$$

$$\beta_i \equiv \beta(S_i)$$

$$\alpha_i \equiv \alpha(S_i)$$

$$\gamma_i \equiv \gamma(S_i)$$

Identity coefficients of the transport matrix:

$$\begin{bmatrix} x(s) \\ x'(s) \end{bmatrix} = \begin{bmatrix} C(s|S_i) & S(s|S_i) \\ C'(s|S_i) & S'(s|S_i) \end{bmatrix} \begin{bmatrix} x_i \\ x_i' \end{bmatrix}$$

$$\begin{aligned} & (s'^2 \gamma_i - 2s'c'\alpha_i + c'^2 \beta_i) x^2 + \\ & 2(-ss'\gamma_i + s'c\alpha_i + sc'\alpha_i - cc'\beta_i) x x' + \\ & (s^2 \gamma_i - 2sc\alpha_i + c^2 \beta_i) x'^2 = \epsilon \end{aligned}$$

- or -

$$\begin{bmatrix} \beta \\ \alpha \\ \gamma \end{bmatrix} = \begin{bmatrix} c^2 & -2sc & s^2 \\ -cc' & (s'c + sc') & -ss' \\ c'^2 & -2s'c' & s'^2 \end{bmatrix} \begin{bmatrix} \beta_i \\ \alpha_i \\ \gamma_i \end{bmatrix}$$

Example: Ellipse Maps:

Drift:
length s

$$\begin{bmatrix} q & s' \\ q' & s' \end{bmatrix} = \begin{bmatrix} 1 & s-s_0 \\ 0 & 1 \end{bmatrix}$$

$$\beta = \beta_0 - 2\alpha_0(s-s_0) + \gamma_0(s-s_0)^2$$

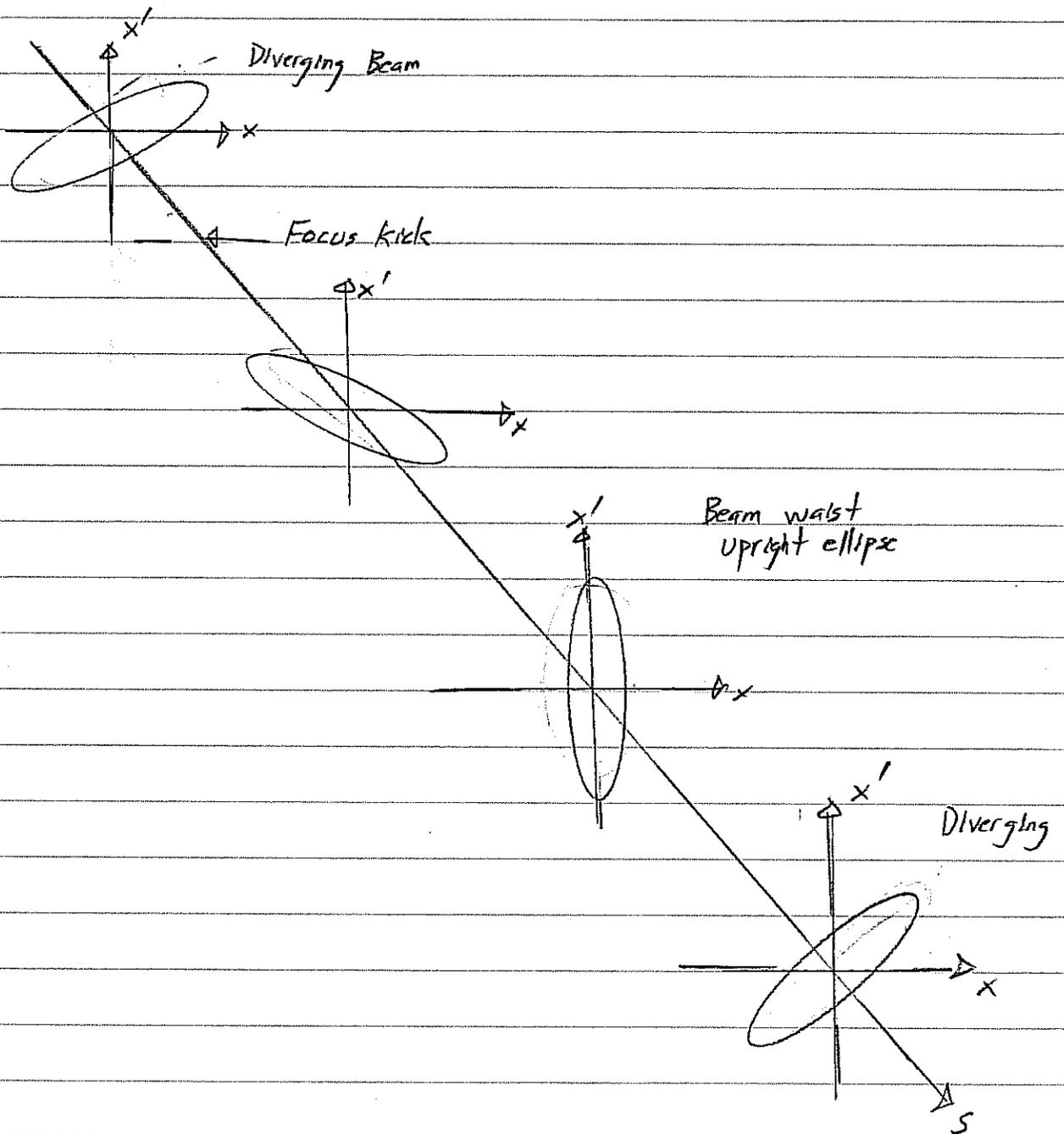
$$\alpha = \alpha_0 - \gamma_0(s-s_0)$$

$$\gamma = \gamma_0$$

Focusing
lense
(d-screen)
kick

$$\begin{bmatrix} q & s' \\ q' & s' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix}$$

$f > 0$ focusing strength.



§ 8 Betatron Formulation of Particle Orbit

The phase amplitude form we originally chose

$$x(s) = A_i W(s) \cos \Psi(s)$$

is not a unique choice and W has unusual dimensions of meters^{1/2}. Due to this and the utility of the Twiss parameters in describing the Courant-Snyder invariant and the orientation of the phase space ellipse on which the particle moves, it is convenient to take an alternative phase-amplitude representation with:

$$x(s) = \sqrt{\epsilon'} \sqrt{\beta(s)} \cos \Psi(s)$$

β -tron function: $\beta(s) \equiv W^2(s)$
 emittance: $\epsilon' = A_i^2 = \text{const.}$
 phase: $\Psi(s) = \Psi_i + \int_{s_i}^s \frac{d\bar{s}}{\beta(\bar{s})} = \Psi_i + \Delta\Psi(s)$

Here, $\beta(s) = W^2$ has dimensions of meters.

Note that the initial phase Ψ_i will differ from the initial phase in the previous phase-amplitude form $x = A_i W \cos(\Psi_0 + \Delta\Psi)$ since $\sqrt{\epsilon'} > 0$ (we choose not to distinguish the Ψ_i for notational simplicity), but the phase advance!

$$\Delta\Psi(s) = \int_{s_i}^s \frac{d\bar{s}}{W^2(\bar{s})} = \int_{s_i}^s \frac{d\bar{s}}{\beta(\bar{s})}$$

is identical in both formulations.

In this case the β -tron function satisfies:

$$\frac{1}{2} \beta(s) \beta''(s) - \frac{1}{4} \beta'^2(s) + \beta^2(s) K(s) = 1$$

$$\beta(s + L_p) = \beta(s)$$

$$\beta(s) > 0.$$

From the particle orbit equation

$$x = \sqrt{\epsilon \beta} \cos \psi,$$

the maximum possible excursion of the particle, $x_m = \text{Max}[|x|]$ occurs where $\cos \psi = \pm 1$ and

$$x_m(s) = \sqrt{\epsilon_m \beta(s)} = \sqrt{\epsilon_m} W(s)$$

$\epsilon_m = \text{max}$
single-particle
emittance.

Since all particles move on nested ellipses, the maximum possible excursion of all particles is given by this expression with the maximum single-particle emittance. This excursion is the minimum possible aperture of the accelerator structures to prevent single particle loss. Resonance effects due to nonlinear focusing fields, space-charge, image charges, imperfect vacuum, and finite particle lifetime all influence the practical aperture choice.

From the equation for w :

$$W''(s) + R(s)W(s) - \frac{1}{W^3(s)} = 0$$

$$W(s+L_p) = W(s) \quad ; \quad W(s) > 0$$

we obtain the equation for the maximum envelope of particle orbits $X_m = \sqrt{\epsilon_m} W$.

$$X_m''(s) + R(s)X_m(s) - \frac{\epsilon_m^2}{X_m^3(s)} = 0$$

$$X_m(s+L_p) = X_m(s)$$

$$X_m > 0$$

Note the similarity of this equation to the statistical envelope equations derived in the intro lectures by J.J. Barnard

- When an extra term is added to include linear space-charge forces this equation is the commonly used KV envelope equation.
- Connection will be covered in more detail later.

Comment:

The use of β for this function along with α, β, γ for Twiss parameters does not often result in confusion with relativistic factors (often denoted β, γ rather than β_0 and γ_0 as used here) because the context usually makes the intent clear. Often relativistic factors are absorbed into definitions of focusing strengths R too, etc.

§9 Momentum Spread Effects

Up to this point, we have analyzed particle dynamics with momentum

$$p_s = m \gamma_b \beta_{bc} = \text{const.}$$

for all particles in a beam slice. In reality, there will be a finite spread of particle momenta in the slice:

$$p_s = p_0 + \delta p$$

$$p_0 = m \gamma_b \beta_{bc} = \text{design momentum}$$

$$\delta p = \text{"off" momentum}$$

Typical values for momentum spread are:

$$\frac{|\delta p|}{p_0} \sim 10^{-2} \text{ to } 10^{-6}$$

in a beam with a single species of particles and a machine with conventional sources and accelerating structures. This spread of momentum can modify particle orbits, particularly in machines with dipole bends where the bending radius of a particle varies strongly with the particle momentum.

To better understand this effect, we analyze the particle equations of motion with leading order dispersive effects:

$$x''(s) + \left[\frac{1}{R(s)^2} \frac{1-\delta}{1+\delta} + \frac{K_x(s)}{(1+\delta)^n} \right] x(s) = \frac{\delta}{1+\delta} \frac{1}{R(s)}$$

$$y''(s) + \frac{K_y(s)}{(1+\delta)^n} y(s) = 0$$

where:

$R(s)$ = Local bend radius

for design momentum p_0

($R \rightarrow \infty$ in straight sections)

$$\frac{1}{R} = \frac{B_y^{\text{dipole}}}{[B\rho]}$$

$$[B\rho] = p_0/q$$

$$\delta \equiv \frac{\Delta p}{p_0} = \text{Fractional momentum error.}$$

$$n = \begin{cases} 1 & ; \text{Magnetic Quadrupoles} \\ 2 & ; \text{Solenoids, Electric Quadrupoles} \end{cases}$$

Here we have neglected:

- Space charge effects: $\phi \rightarrow 0$

- Nonlinear applied focusing: \vec{E}^a, \vec{B}^a contain only linear focusing terms.

- Acceleration: $p_0 = m \gamma_b \beta_b = \text{const.}$

Also, the bends are set from the design particle orbit (see §1)

Terms resulting from the momentum spread can be lumped into two classes, chromatic and dispersive.

Chromatic terms : present in both x- and y- equations resulting from applied focusing fields

$$R_x \cdot X_i \rightarrow \frac{R_x X}{(1+\delta)^n} \quad \text{where } R_{x,y} \text{ remains defined by the design momentum.}$$

$$R_y \cdot y \rightarrow \frac{R_y Y}{(1+\delta)^n}$$

Particle momentum changes vary the focal strength of optics and will thereby modify the phase space trajectory of particles.

- Generally not important (corrections not large) except for very small focal spots when the beam is focused onto a target or when momentum spreads are large.
- Lectures by J.J. Barnard will overview consequences of this effect in final focus optics. We will neglect these terms here.

Dispersive terms : only in the x-equation. results from dipole bends in x

Result from particles being bent at different radii when the momentum deviates from the design value. Dispersive terms contain the bend radius R :

$$\frac{1}{R^2(s)} \frac{1-\delta}{1+\delta} x(s) \quad ; \quad \frac{\delta}{1+\delta} \frac{1}{R(s)}$$

Generally, bend radii R are large and δ is small, and we can take to leading order!

$$\left[\frac{1-\delta}{R^2} + R_x \right] x \approx R_x x$$

$$\frac{\delta}{1+\delta} \frac{1}{R} \approx \frac{\delta}{R}$$

and the equations become:

$$\begin{aligned} x''(s) + R_x(s)x(s) &= \frac{\delta}{R(s)} \\ y''(s) + R_y(s)y(s) &= 0 \end{aligned}$$

At this level of approx.,
no change in y -equation

Generally, the x -equation is solved for periodic lattices by exploiting the linear structure of the equation and resolving:

$$x = x_p + x_h$$

\uparrow \uparrow
 Particular Sol. Homogeneous sol.

where

x_h is the general solution to:

$$x_h''(s) + R_x(s) x_h(s) = 0$$

and x_p is the periodic solution to:

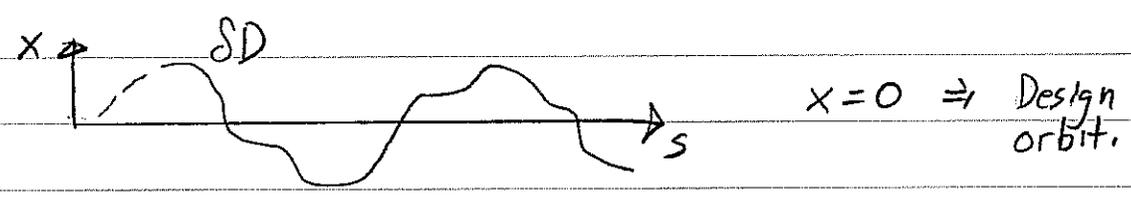
$$x_p = \delta \cdot D$$

$$D''(s) + R_x(s) D(s) = \frac{1}{R(s)}$$

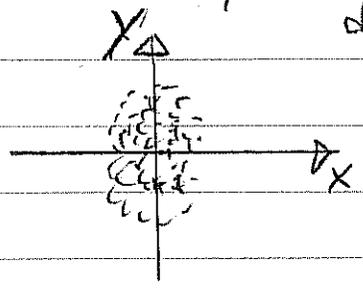
$$D(s + L_p) = D(s)$$

This convenient resolution can always be made.

D is called the dispersion function and $\delta \cdot D$ measures the offset of the particle orbit with respect to the design orbit.

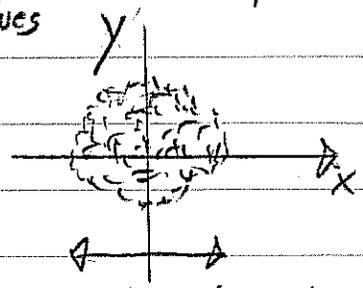


Bundle of particles $D=0$



distribution of δ values

Bundle of particles $D \neq 0$



Note: $[D] = \text{meters}$
 $\Rightarrow \delta \cdot D$ gives orbit offset in meters.

// Example - Continuous Focusing in a continuous bend.

$$\rho_x(s) = \rho_{p0}^2 = \text{const.}$$

$$R(s) = R = \text{const.}$$

$$\Rightarrow D'' + \rho_{p0}^2 D = \frac{1}{R} \Rightarrow D = \frac{1}{\rho_{p0}^2 R}$$

In periodic lattices D will oscillate with (approx.) average value:

$$D \sim \frac{1 L_p^2}{\rho_{p0}^2 R} //$$

Many rings are designed to focus the dispersion function to small values in straight sections even though the lattice has strong bends. Quadrupole triplet focusing lattices often are employed for rings since the optics allow enough freedom to do this. This is desirable since it allows smaller beam size at locations where $D \approx 0$ and these locations can be used to more easily insert and extract (kick) the beam into and out of the ring.

§10 Acceleration and Normalized Emittance

If the beam is accelerated, Hill's equation for the single-particle orbit becomes:

$$x''(s) + \frac{(\gamma_b \beta_b)'}{(\gamma_b \beta_b)} x'(s) + K_x(s) x(s) = -\frac{g}{m \gamma_b^3 \beta_b^2 c^2} \frac{\partial \phi}{\partial x}$$

↑
Acceleration term

Neglects:

- Nonlinear applied focusing: \vec{E}^a, \vec{B}^a contain only linear focusing terms
- Momentum spread effects.

Also, implicit in this equation, is that any energy variations in $K_x(s)$ are compensated for in focusing optics if the lattice is periodic, or otherwise incorporated in the s -dependence.

In typical applications the changes induced by the acceleration term are slow and the fractional energy change induced by individual acceleration cells is small (except possibly near the injector).

The acceleration term

$$\frac{(\gamma_b \beta_b)'}{(\gamma_b \beta_b)} > 0 \Rightarrow \text{damped harmonic oscillator equation in } x.$$

and oscillations in the particle orbit will damp. $\gamma_b \beta_b$ will be a function of s prescribed by the acceleration schedule of the machine - derived

consistently from machine gap structures in the lattice, practical accelerating field breakdown limits. Even with acceleration, we will find that the particle orbit has a Courant-Snyder invariant (normalized emittance) if coordinates are appropriately chosen to compensate for the acceleration induced damping of the particle phase-space area.

Transform:

$$\tilde{x} \equiv \sqrt{\gamma_b \beta_b'} x$$

"Guess" can be motivated qualitatively based on conjugate variable arguments - see Intro. lecture J.J. Barnard

Then

$$\begin{aligned} x &= \frac{\tilde{x}}{\sqrt{\gamma_b \beta_b'}} & \Rightarrow & \tilde{x} = \sqrt{\gamma_b \beta_b'} x \\ x' &= \frac{\tilde{x}'}{\sqrt{\gamma_b \beta_b'}} - \frac{1}{2} \frac{(\gamma_b \beta_b)'}{(\gamma_b \beta_b)^{3/2}} \tilde{x} & \tilde{x}' &= \sqrt{\gamma_b \beta_b'} x' + \frac{1}{2} \frac{(\gamma_b \beta_b)'}{\sqrt{\gamma_b \beta_b}} x \\ x'' &= \frac{\tilde{x}''}{\sqrt{\gamma_b \beta_b'}} - \frac{(\gamma_b \beta_b)'}{(\gamma_b \beta_b)^{3/2}} \tilde{x}' + \frac{3}{4} \frac{(\gamma_b \beta_b)'^2}{(\gamma_b \beta_b)^{5/2}} \tilde{x} - \frac{1}{2} \frac{(\gamma_b \beta_b)''}{(\gamma_b \beta_b)^{3/2}} \tilde{x} \end{aligned}$$

substituting these factors, the particle equation of motion with acceleration becomes:

$$\tilde{x}'' + \left[K_x + \frac{1}{4} \frac{(\gamma_b \beta_b)'^2}{(\gamma_b \beta_b)^2} - \frac{1}{2} \frac{(\gamma_b \beta_b)''}{(\gamma_b \beta_b)} \right] \tilde{x} =$$

Note factor of $\frac{1}{\gamma_b \beta_b}$ difference from lab frame in the space charge term $\rightarrow -\frac{q}{m \gamma_b^2 \beta_b c^2} \frac{\partial \phi}{\partial \tilde{x}}$

To apply this equation to beams with finite space-charge, the Poisson equation must also be transformed:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \phi = -\frac{\rho}{\sqrt{\epsilon_0}} \quad \rho = \text{local charge density}$$

$$\begin{aligned} \frac{\partial^2}{\partial x^2} &= \left(\frac{\partial \tilde{x}}{\partial x} \frac{\partial}{\partial \tilde{x}} \right) \left(\frac{\partial \tilde{x}}{\partial x} \frac{\partial}{\partial \tilde{x}} \right) \\ &= \gamma_b \beta_b \frac{\partial^2}{\partial \tilde{x}^2} \end{aligned}$$

y-direction transforms the same:

$$\left(\frac{\partial^2}{\partial \tilde{x}^2} + \frac{\partial^2}{\partial \tilde{y}^2} \right) \phi = -\frac{\rho}{\sqrt{\gamma_b \beta_b \epsilon_0}}$$

Denote:

$$\begin{aligned} \tilde{\phi} &= \gamma_b \beta_b \phi \\ \Rightarrow \left(\frac{\partial^2}{\partial \tilde{x}^2} + \frac{\partial^2}{\partial \tilde{y}^2} \right) \tilde{\phi} &= -\frac{\rho}{\sqrt{\epsilon_0}} \end{aligned}$$

In terms of $\tilde{\phi}$ the particle eqn of motion is:

$$\tilde{x}'' + \left[k_x + \frac{1}{4} \frac{(\gamma_b \beta_b)'}{(\gamma_b \beta_b)^2} - \frac{1}{2} \frac{(\gamma_b \beta_b)''}{(\gamma_b \beta_b)} \right] \tilde{x} = -\frac{\rho}{m \gamma_b^3 \beta_b^2 c^2} \frac{\partial \tilde{\phi}}{\partial \tilde{x}}$$

← Restores the "usual" form of the space-charge term.

The transformed equation can be expressed as

$$\tilde{x}'' + \tilde{K}_x \tilde{x} = -\frac{q}{m \gamma_b^3 \beta_b^2 c^2} \frac{\partial \phi}{\partial x}$$

$$\tilde{K}_x \equiv K_x + \frac{1}{4} \frac{(\delta_b \beta_b)'^2}{(\delta_b \beta_b)^2} - \frac{1}{2} \frac{(\delta_b \beta_b)''}{(\delta_b \beta_b)}$$

where \tilde{K}_x is a corrected focusing constant including additional acceleration terms beyond those already contained in K :

$$K_x = \begin{cases} \pm K_g = \frac{qG}{m \delta_b \beta_b^2 c^2} & ; G = \text{Electric Quadrupole Gradient} \\ \pm K_g = \frac{qG}{m \delta_b \beta_b c} & ; G = \text{Magnetic Quadrupole Gradient} \\ \frac{\omega_c^2}{4 \gamma_b^2 \beta_b^2 c^2} = \frac{q B_z^2}{4 m \delta_b^2 \beta_b^2 c^2} & ; B_z^2 = \text{Solenoidal Magnetic Field.} \end{cases}$$

Note that the transformed equation has the same form as usual Hill's equation with no acceleration when space-charge is neglected ($\phi \rightarrow 0$).

$$x'' + K_x x = 0 \iff \tilde{x}'' + \tilde{K}_x \tilde{x} = 0$$

Same form

Therefore, all previous analysis on Courant-Snyder invariants in $x-x'$ phase space can be immediately applied to the $\tilde{x}-\tilde{x}'$ phase-space for an accelerating beam.

$$\tilde{x} \equiv x - \frac{1}{\beta_b} \int \frac{v}{c} dx = \text{const.}$$

$\tilde{\phi} \rightarrow 0$ Courant-Snyder Invariant

$$\left(\frac{\tilde{x}}{\tilde{w}_x} \right)^2 + \left(\tilde{w}_x \tilde{x}' - \tilde{w}_x' \tilde{x} \right)^2 = \tilde{\epsilon}^2 = \text{const}$$

$$\tilde{w}_x'' + \tilde{K} \tilde{w}_x - \frac{1}{\tilde{w}_x^3} = 0$$

$$\tilde{w}_x(s+L_p) = \tilde{w}_x(s)$$

where:

$$\pi \tilde{\epsilon}^2 = \text{Area traced by orbit} = \text{const.}$$

in $\tilde{x}-\tilde{x}'$ phase-space

It is instructive to relate these acceleration variable " $\tilde{\epsilon}$ " emittances to the instantaneous lab-frame $x-x'$ phase-space area.

Phase space areas are related as

$$d\tilde{x} \otimes d\tilde{x}' = |J| dx \otimes dx'$$

where J is the Jacobian

$$J = \det \begin{vmatrix} \frac{\partial \tilde{x}}{\partial x} & \frac{\partial \tilde{x}'}{\partial x} \\ \frac{\partial \tilde{x}'}{\partial x} & \frac{\partial \tilde{x}}{\partial x} \end{vmatrix} = \det \begin{vmatrix} \sqrt{\gamma_b \beta_b} & 0 \\ \frac{1}{2} \frac{(\gamma_b \beta_b)'}{\sqrt{\gamma_b \beta_b}} & \sqrt{\gamma_b \beta_b} \end{vmatrix}$$

$$= \gamma_b \beta_b$$

Thus:

$$d\tilde{x} \otimes d\tilde{x}' = \gamma_b \beta_b dx \otimes dx'$$

Based on this area transform, if we define an (instantaneous) area of the orbit traced in $x-x'$ to be $\pi \cdot \varepsilon$ (regular emittance):

$$\tilde{\varepsilon} = \gamma_b \beta_b \varepsilon = \text{const.}$$

$$\equiv \text{"Normalized" Emittance.} \equiv \varepsilon_n$$

The $\gamma_b \beta_b$ factor compensates for the acceleration induced damping of particle oscillations.

§10 Appendix A.

Accelerating Fields and Calculation of $\gamma_b \beta_b$

The equation of motion with acceleration was derived by approximating:

$$\frac{d}{dt} \left(\gamma \frac{dx}{dt} \right) = \frac{q}{m} E_x^a - \frac{q}{m} v_z^a B_y^a + \frac{q}{m} v_y^a B_z^a$$

Using:

Neglect Solenoidal field for simplicity.

$$\frac{d}{dt} \left(\gamma \frac{dx}{dt} \right) \approx \gamma_b \beta_b^2 c^2 \left(\ddot{x} + \frac{(\gamma_b \beta_b)'}{(\gamma_b \beta_b)} \dot{x} \right)$$

Changes in $\gamma_b \beta_b$ are calculated from the longitudinal particle equation

$$\frac{d}{dt} \left(\gamma \frac{dz}{dt} \right) = \frac{q}{m} E_z^a + \frac{q}{m} v_x^a B_y^a - \frac{q}{m} v_y^a B_x^a$$

Weak change in particle longitudinal energy, neglect.

with:

$$1) \quad \frac{d}{dt} \left(\gamma \frac{dz}{dt} \right) \approx c^2 \beta_b (\gamma_b \beta_b)'$$

$$2) \quad \frac{q}{m} E_z^a \approx \left. \frac{-q}{m} \frac{\partial \phi^a}{\partial z} \right|_{r=0}$$

where ϕ^a is the quasi-static approx. accelerating potential from an accelerating "gap" (induction, RF, etc.)
 The acceleration also produces \perp focusing fields:

$$E_x^a \approx E_x^a \Big|_{\text{Focusing Optics}} - \frac{\partial \phi^a}{\partial x} \Big|_{\text{from Acceleration}} \perp \text{ Focusing}$$

The longitudinal equation then becomes:

$$\beta_b (\gamma_b \beta_b)' \approx \frac{-g}{4mc^2} \frac{\partial \phi^g}{\partial z} \Big|_{r=0}$$

But:

$$\gamma_b' = \left(\frac{1}{\sqrt{1-\beta_b^2}} \right)' = \gamma_b^3 \beta_b \beta_b'$$

$$\Rightarrow \beta_b (\gamma_b \beta_b)' = (1 - \gamma_b^2 \beta_b^2) \gamma_b \beta_b \beta_b' = \gamma_b^3 \beta_b \beta_b' = \gamma_b'$$

and the longitudinal equation can be integrated!

$$\gamma_b' = \frac{-g}{4mc^2} \frac{\partial \phi^g}{\partial z} \Big|_{r=0}$$

$$\Rightarrow \boxed{\gamma_b + \frac{g}{4mc^2} \phi^g(r=0, z) = \text{const.}}$$

This result can be used to cast terms occurring in the transverse equation of motion as

$$x'' + \frac{(\gamma_b \beta_b)'}{(\gamma_b \beta_b)} x' \approx \frac{g}{m \gamma_b \beta_b^2 c^2} E_x^g \quad \begin{array}{l} \text{Applied Focus} \\ \text{Electric Quads etc} \end{array} \quad \frac{-g}{m \gamma_b \beta_b c} B_y^g \quad \begin{array}{l} \text{Magnetic} \\ \text{FOCUS} \end{array}$$

$$\frac{-g}{m \gamma_b \beta_b^2 c^2} \frac{\partial \phi^g}{\partial x} \quad \begin{array}{l} \text{Transverse foc} \\ \text{From Accel. gap.} \end{array}$$

In the quasi-static approximation, the axisymmetric field of an accelerating gap can be expanded as: (see J.J. Barnard, Intro lectures and Reiser Sec. 3.3):

$$E_z^g = -\frac{\partial \phi^g}{\partial z}$$

$$\phi^g = V(z) - \frac{1}{4} \frac{\partial^2 V(z)}{\partial z^2} (x^2 + y^2) + \frac{1}{64} \frac{\partial^4 V(z)}{\partial z^4} (x^2 + y^2)^2 + \dots$$

where $V(z)$ is the on-axis potential due to gap accelerating structures.

Using these results it is possible to cast certain acceleration terms in more convenient forms - both in the equation for x and the transformed equation for \tilde{x} . An example of this can be found in the intro. lectures by J.J. Barnard.

References

The course textbook Reiser provides a solid introduction to many of the topics covered. For further reading including more advanced treatments, we suggest the following books:

Floquet's Theorem, Courant-Snyder Invariants, Hill's Equation
* H. Wiedemann, "Particle Accelerator Physics, ^{Dispersion Function...}
Basic Principles and Linear Beam Dynamics."
Springer Verlag, Berlin, 1993.

* Highly Recommended

Resonances and Solenoidal Focusing

H. Wiedemann, "Particle Accelerator Physics II,
Nonlinear and Higher-Order Beam Dynamics."
Springer Verlag, Berlin, 1995.

D.A. Edwards and M.J. Syphers, "An Introduction
to the Physics of High Energy Accelerators,"
John Wiley & Sons, Inc., New York, 1993.

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some additional references:

Particle equations of motion with bends and momentum spread:

D.A. Edwards and M.J. Syphers, "An Introduction to the Physics of High Energy Accelerators," John Wiley & Sons, New York, 1993.

Original classic paper on strong alternating gradient focusing

E.D. Courant and H.S. Snyder, "Theory of the Alternating Gradient Synchrotron" *Annals of Physics* 3, 1 (1958).

More mathematical treatment of transfer maps and stability. A standard, high quality reference

A. Dragt, "Lectures on Nonlinear Orbit Dynamics," in "Physics of High Energy Accelerators," edited by R.A. Carrigan, F.R. Hudson, and M. Month. (AIP Conf. Proc. No. 87. (AIP, New York, 1982) p. 147.

Recent review containing related information on phase advances, orbit stability, etc.

S.M. Lund and B. Bukh, "Stability Properties of the Transverse Envelope Equations Describing Intense Ion Beam Transport," to appear *Phys. Rev. Special Topics - Accelerators and Beams*, 2004.